Detailed description of the two-ion Ramsey experiment

For reference, we will use a phase convention such that for single qubit rotations,

\[ | \downarrow \rangle \rightarrow -ie^{-i\phi} \sin(\theta/2) | \uparrow \rangle + \cos(\theta/2) | \downarrow \rangle \]

and

\[ | \uparrow \rangle \rightarrow \cos(\theta/2) | \uparrow \rangle - ie^{i\phi} \sin(\theta/2) | \downarrow \rangle, \]

where \( \phi \) is the angle of the vector (in the \( xy \)-plane) about which the Bloch vector is rotated and \( \theta \) is the angle of rotation. Similarly, for sideband transitions, we will assume

\[ | \downarrow \rangle \rightarrow -ie^{-i\phi} \sin(\theta/2) | \uparrow \rangle + \cos(\theta/2) | \downarrow \rangle | n \rangle_m \]

and

\[ | \uparrow \rangle \rightarrow \cos(\theta/2) | \uparrow \rangle - ie^{i\phi} \sin(\theta/2) | \downarrow \rangle | n \rangle_m. \]

Starting from the initial state \( \psi_0 = | \downarrow \rangle_S | \downarrow \rangle_L | 0 \rangle_m \), we perform the two-ion Ramsey experiment by first applying an optical CAR \( \pi/2 \)-pulse on the \( ^{27}\text{Al}^+ \) ion, leading to

\[ \psi_0 \rightarrow \psi_1 = \left| \downarrow \rangle_S - ie^{-i\phi_0} | \uparrow \rangle_S \right| \downarrow \rangle_L | 0 \rangle_m \]

\[ = \left| \downarrow \rangle_S | 0 \rangle_m - ie^{-i\phi_0} | \uparrow \rangle_S | 0 \rangle_m \right| \downarrow \rangle_L, \]

where \( \phi_0 \) is the (optical) phase at the position of the ion. This electronic superposition is then coherently mapped to a motional superposition on the transfer mode by applying a RSB \( \pi \)-pulse on the \( ^{27}\text{Al}^+ \) ion, which gives

\[ \psi_1 \rightarrow \psi_2 = \left| \downarrow \rangle_S | 0 \rangle_m - e^{-i(\phi_0-\phi_1)} | \downarrow \rangle_S | 1 \rangle_m \right| \downarrow \rangle_L \]

\[ = \left| \downarrow \rangle_S | \downarrow \rangle_L \left[ | 0 \rangle_m - e^{-i(\phi_0-\phi_1)} | 1 \rangle_m \right] \right| \downarrow \rangle_L. \]

In the subsequent free evolution with duration \( t_d \), the transfer mode superposition state acquires a phase shift \( \Delta \phi = \omega t_d \) where \( \omega \) is the transfer mode frequency, so that

\[ | 0 \rangle_m - e^{-i(\phi_0-\phi_1)} | 1 \rangle_m \rightarrow | 0 \rangle_m - e^{-i(\phi_0-\phi_1+\Delta \phi)} | 1 \rangle_m. \]

Mapping this phase-evolved superposition onto the \( ^{9}\text{Be}^+ \) ion is accomplished by applying a RSB \( \pi \)-pulse on the \( ^{9}\text{Be}^+ \) ion

\[ \psi_2 \rightarrow \psi_3 = \left| \downarrow \rangle_S | \downarrow \rangle_L | 0 \rangle_m + ie^{-i(\phi_0-\phi_1+\Delta \phi+\phi_2)} | \uparrow \rangle_L | 0 \rangle_m \right| \downarrow \rangle_L \]

\[ = \left| \downarrow \rangle_S | 0 \rangle_m \left[ | \downarrow \rangle_L + ie^{-i(\phi_0-\phi_1+\Delta \phi+\phi_2)} | \uparrow \rangle_L \right] \right| \downarrow \rangle_L. \]
To complete the Ramsey experiment, we apply the second (radio-frequency) Ramsey $\pi/2$-pulse on the $^{9}\text{Be}^{+}$ ion and obtain
\[
\psi_3 \rightarrow \psi_4 = |\downarrow\rangle_S |0\rangle_m \left[ (1 + e^{-i(\phi_0 - \phi_1 + \Delta \phi + \phi_2 - \phi_3)}) |\downarrow\rangle_L 
+ i e^{-i\phi_3} (e^{-i(\phi_0 - \phi_1 + \Delta \phi + \phi_2 - \phi_3)} - 1) |\uparrow\rangle_L \right].
\]

We detect the population $P_{\downarrow,L} = \cos^2 \left( \frac{\phi_0 - \phi_1 + \phi_2 - \phi_3 + \Delta \phi}{2} \right)$ of the state $|\downarrow\rangle_L$, which depends only on $\Delta \phi$ and the differential phases of the carrier and sideband pulses applied to each ion. The carrier and sideband pulses are derived from the same laser beam with the use of different frequencies applied to a frequency modulator. The signal $P_{\downarrow,L}$ is therefore independent of the optical phase for the $^{27}\text{Al}^+$ transition and the Raman phase on the $^{9}\text{Be}^+$ transition as long as these phases do not change between the carrier and sideband pulses. Moreover, for every experiment, the phase differences $\phi_0 - \phi_1$ and $\phi_2 - \phi_3$ are made to be the same, so that the evolution of the Ramsey fringes depends only on $\Delta \phi$. 

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