Supporting Online Material for

Metal-Insulator Transition in Disordered Two-Dimensional Electron Systems
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Definition of the model

The renormalization of the parameters of the disordered electron gas is best described by the matrix non-linear sigma model \((1, 2)\):

\[
S[Q] = \frac{\pi \nu}{8} \int \left[ \text{Tr} \left( \nabla^2 Q \right)^2 - 4 \pi \nu \text{Tr}(\tilde{\epsilon} Q) + Q^2 \hat{\Gamma} Q \right].
\] \( (1) \)

The matrix \(Q\) satisfies the constraints: \(Q^2 = 1\), \(Q = Q^\dagger\), and \(\text{Tr} Q = 0\). The components of \(Q\) are defined as \(Q_{ij,\alpha\beta}^{nm}\), where \(n\) and \(m\) are the fermionic energy indices with \(\epsilon_n = (2n + 1)\pi T\); \(i, j\) are the replica indices and \(\alpha, \beta\) are the spin and valley indices. The trace is taken over all these variables. The parameter \(\nu\) is the density of states of a single spin and valley species and \(D\) is the diffusion constant. The energy matrix \(\hat{\epsilon} = \epsilon_n \delta_{nm} \delta_{ij} \delta_{\alpha\beta}\) and \(z\) is the frequency renormalization factor that determines the relative scaling of the frequency with respect to the length scale; \(z = 1\) for free electrons. Depending on the spin and valley structure the two-particle interaction \(\hat{\Gamma}\) is split in terms of the two Fermi-liquid amplitudes: \(\hat{\Gamma}_{\alpha\beta}^{\alpha'\beta'} = \Gamma_1 \delta_{\alpha\alpha'} \delta_{\beta\beta'} - \Gamma_2 \delta_{\alpha\beta} \delta_{\alpha'\beta'}\). It is assumed that the interactions couple electrons in different valleys but do not mix them. Inter-valley scattering processes due to the disorder are also neglected. In addition, there exists the interaction \(\Gamma_c\) in the Cooperon channel \((3)\).

The \(Q\)-matrix is parameterized so that the constraints are automatically satisfied; we choose the parameterization \(Q = (1 - q/2) \Lambda (1 - q/2)^{-1}\) \((4)\), where the matrices \(q\) and \(\Lambda\) satisfy: \(q^\dagger = -q\), \(\Lambda^2 = 1\), \(\text{Tr} \Lambda = 0\), and \(\Lambda q = -q \Lambda\). The elements of \(q\) for a given set of replica and energy indices can be expressed \((5)\):

\[
q_{ij}^{nm}(k) = \begin{pmatrix} 0 & b_k \\ -b_k^\dagger & 0 \end{pmatrix}_{nm}^{ij}, \quad b_k = \begin{pmatrix} d_k \\ sc_k^s \\ sd_k^s \\ s \end{pmatrix},
\] \( (2) \)

in terms of the matrix \(b\), which in turn can be expressed in terms of the operators \(d_k\) and \(c_k\). The constant matrix \(s = -s^T\) and \(s^2 = -1\). Physically, the \(d\) and \(c\)-fields are the creation and annihilation operators of the low energy Diffuson and Cooperon modes:

\[
\langle d_k d_{-k}^\dagger \rangle = \langle c_k c_{-k}^\dagger \rangle = \frac{2}{\pi \nu} \frac{1}{Dk^2 + z\omega},
\] \( (3) \)

where \(\omega = \epsilon_n - \epsilon_m > 0\), with the energies \(\epsilon_n > 0\) (particle) and \(\epsilon_m < 0\) (hole) lying on opposite sides of the Fermi surface.
Technicalities of the Renormalization Group

Scaling variables: The relevant scaling variables can be delineated by writing the action in Eq. (1) as:

\[ S[Q] = \frac{\pi}{8} \int (\nu z) \left[ \text{Tr} \left( D/z \right) (\nabla Q)^2 - 4\text{Tr} (\dot{Q}) + Q\dot{\gamma} Q \right]. \]  

(4)

It follows that: (i) the interaction amplitudes appear in the combinations \( \gamma = \Gamma/z \).
(ii) Since the pre-factor scales as \( z\nu \), the parameter \( z \) has the natural interpretation of the density of states of the underlying diffusion modes. Unlike the case of free electrons, the coefficient \( z \) changes in the process of renormalization leading to the renormalization of the density of states of these modes. The Einstein relation for the diffusion coefficient and the conductivity reads:

\[ \sigma/e^2 = 2n_v(\nu z)(D/z). \]

Hence, the dimensionless resistance parameter per valley \( t = 1/(2\pi)^2 \nu D \). It follows that the relevant set of scaling variables are \( t, \gamma \) and \( z \).

Separation of the diffusion modes into fast and slow variables: The renormalization group method is used to study the logarithmic corrections which appear due to the electron interactions. To this end, it is necessary to separate the fast and the slow variables and regularize the divergent integrals in an invariant manner. The standard procedure, as used in the vector \( \sigma \)-model (6), involves representing the \( Q \)-matrix in terms of fast \( q_0 \) and slow \( \tilde{q} \) variables as:

\[ Q = (1 - \tilde{q})(1 - q_0)\Lambda(1 - q_0)^{-1}(1 - \tilde{q})^{-1}. \]

(5)

The above expression for \( Q \) is substituted into \( S[Q] \) and the fast variables are expanded up to order \( q_0^4 \). This ensures that all terms up to order \( t^2 \) are retained, as can be seen by re-scaling \( q_0 = \sqrt{t}q_0 \), so that \( q_0^4 = \mathcal{O}(t^2) \). This defines a field theory with nonlinear interaction terms that can be analyzed diagrammatically.

Examples of diagrams that contribute to the renormalization of \( t \), i.e., the term proportional to \( Dk^2Q_kQ_{-k} \) in \( S[Q] \) (the energy indices are suppressed for brevity), are shown in Fig. 1. The diagrams are constructed by bringing together various terms in the expansion of \( S[Q] \), as explained in the caption. Only those diagrams are retained which contain two independent momentum integrations. Similar diagrams exist for the renormalization of the amplitudes \( \gamma \) and \( z \). It is obvious from these diagrams that the number of combinations at \( t^2 \)-order (two-loop order) are enormous. One can, however, see that by constraining the \( \dot{\gamma} \)-vertices to be attached to closed-loops only limits somewhat the number of combinations, allowing for a consistent approximation scheme in which only those diagrams which survive in...
Figure 1: Two examples of diagrams that contribute to the renormalization of the parameter $t$. The solid black and red lines indicate particle-hole lines, respectively. Each cluster of stripes indicate a Diffusion (black) or a Cooperon (red) propagator (defined in Eq. (3)) obtained by contracting the fast variable $\langle \tilde{q}q_0^2 \rangle$. The wiggly lines correspond to the interaction vertex $\tilde{\gamma}_2$. Fig. (a) is constructed by bringing together the terms $\nabla^2 (\tilde{q}q_0^3) \times (\tilde{q}q_0^2 \tilde{\gamma}_2 q_0^2)$, and Fig. (b) is constructed out of the terms $\nabla^2 (\tilde{q}q_0^3) \times (q_0^2 \tilde{\gamma}_2 q_0^2) \times (\tilde{q}q_0^2 \tilde{\gamma}_2 q_0^2)$. Note that each wiggly line ($\tilde{\gamma}_2$ vertex) is attached to a closed-loop and that both diagrams contain two independent momentum integrations leading to corrections of order $t^2$.

the $n_v \rightarrow \infty$ limit are retained. The divergent integrals over the fast variables are evaluated by dimensional regularization in the standard way (4, 7, 8).

References