Supplementary Materials for

Roton-Type Mode Softening in a Quantum Gas with Cavity-Mediated Long-Range Interactions

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Roton-type mode softening in a quantum gas with cavity-mediated long-range interactions

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We give details of the experimental setup and provide a theoretical description of the dispersively coupled condensate-cavity system. An effective Hamiltonian for the cavity-mediated long-range interaction is derived. Based on a mean-field description, we numerically calculate the steady state of the system including contact interactions and the transverse lattice potential. We deduce in a Bogoliubov approach the energy of collective excitations, which softens towards the critical point. The time evolution of the system during probing is derived and a diverging response to density perturbations at the phase transition is found.

Experimental details

The atoms are prepared in the hyperfine state \((F, m_F) = (1, -1)\) with respect to a quantization axis pointing along the cavity axis, where \(F\) is the total angular momentum and \(m_F\) the magnetic quantum number. The transverse pump laser with wavelength \(\lambda = 784.5\, \text{nm}\) is linearly polarized along the \(y\)-axis (see Fig. 1 in the main text) and off-resonantly drives, via the atoms, two degenerate cavity TEM\(_{00}\) modes with circular polarizations \(\epsilon_+\) and \(\epsilon_-\). The ratio
of the corresponding two-photon Rabi frequencies is given by $\eta_+/\eta_- = 3.26/1.25$, where all allowed dipole transitions in the $D_1$ and $D_2$ lines of $^{87}\text{Rb}$ have been taken into account. The maximum dispersive shift of the two cavity modes induced by a single maximally coupled atom is $U_0^+ = 2\pi \times 87\ \text{Hz}$ and $U_0^- = 2\pi \times 33\ \text{Hz}$, respectively. The transverse pump laser induces an optical lattice potential along the $z$-axis with periodicity of $\lambda/2$. Its depth $V_p$ is calibrated using Raman-Nath diffraction (31) and takes for our experimental parameters a value of $3E_r$ at the critical point, with recoil energy $E_r = \frac{\hbar^2 k^2}{2m}$, wavevector $k = 2\pi/\lambda$, and atomic mass $m$. In the theoretical analysis, the Gaussian envelopes of the pump and cavity fields along the transverse directions (32) are effectively taken into account by weighted averages of $V_p$, $\eta_\pm$ and $U_0^\pm$ over the spatial extent of the atomic cloud. The Thomas-Fermi radii of the condensate in the external harmonic trapping potential are given by $(R_x, R_y, R_z) = (3.5, 18.3, 3.7)\ \mu\text{m}$, assuming an atom number of $N = 1.65 \times 10^5$.

The length (176 $\mu\text{m}$) of the cavity is actively stabilized using a laser with a wavelength of 830 nm, which is referenced onto the transverse pump laser. The depth of the resulting intracavity lattice potential was measured to be 0.04(2)$E_r$, and is neglected in the theoretical analysis. Transverse pump light and cavity probe light propagate through independent optical fibers, resulting in a variation of their relative phase $\varphi$ between different experimental runs. The cavity output light is monitored on a single-photon counting module with an overall detection efficiency of intracavity photons of 4(1)%. For the data taken in the normal phase, the critical pump power $P_{cr}$ was deduced from the intracavity photon number monitored during independent sweeps across the phase transition.

**Theoretical description of the coupled condensate-cavity system**

*The coupled condensate-cavity system is described in a many-body formalism following Ref. (33, 34). By adiabatically eliminating the fast cavity field dynamics, we derive an effective Hamilto-*
nian, which describes the long-range atom-atom interaction.

After adiabatically eliminating the electronically excited states, the transversally driven condensate-cavity system is described by the many-body Hamiltonian \( \hat{H} = \hat{H}_c + \hat{H}_a + \hat{H}_{a-c} \), where

\[
\hat{H}_c = -\hbar \Delta_c \hat{a} \hat{a}^\dagger \\
\hat{H}_a = \int d^3r \left[ \frac{\mathbf{P}^2}{2m} + V_p \cos^2(kz) + \frac{g}{2} \hat{\Psi}^\dagger(\mathbf{r}) \hat{\Psi}(\mathbf{r}) \right] \hat{\Psi}(\mathbf{r}) \\
\hat{H}_{a-c} = \int d^3r \left[ \hbar \eta \cos(kx) \cos(kz) (\hat{a} + \hat{a}^\dagger) + \hbar U_0 \cos^2(kx) \hat{a} \hat{a}^\dagger \right] \hat{\Psi}(\mathbf{r}),
\]

with bosonic atomic field operator \( \hat{\Psi}(\mathbf{r}) \) and photon operators \( \hat{a} \) and \( \hat{a}^\dagger \). To keep the notation simple we describe only one of the two circularly polarized TEM\(_{00}\) cavity modes. Final results will be given for the case of two cavity modes.

In equation (S1), \( \hat{H}_c \) describes the dynamics of a single TEM\(_{00}\) cavity mode with spatial mode function \( \cos(kx) \), whose frequency \( \omega_c \) is detuned by \( \Delta_c = \omega_p - \omega_c \) from the pump laser frequency \( \omega_p \). The term \( \hat{H}_a \) captures the atomic evolution in the transverse optical lattice potential with depth \( V_p \), including contact interactions with strength \( g = \frac{4 \pi \hbar^2 a}{m} \), where \( a \) denotes the s-wave scattering length. The interaction between the atoms and the pump and cavity light fields is governed by \( \hat{H}_{a-c} \). The first term describes light scattering between pump and cavity field at a rate which is determined by the maximum two-photon Rabi frequency \( \eta \). The second term accounts for the dispersive shift of the cavity resonance frequency with light-shift \( U_0 \) of a single maximally coupled atom.

As the cavity field reaches a steady-state on a time scale fast compared to atomic motion, its equation of motion can be formally solved, yielding

\[
\hat{a} = \frac{\eta \hat{\Theta}}{(\Delta_c - U_0 \mathcal{B}) + i\kappa} \tag{S2}
\]

with the cavity field decay rate \( \kappa = 2\pi \times 1.25 \text{ MHz} \). Due to Bragg scattering of pump light, the intracavity field amplitude is proportional to the order parameter \( \hat{\Theta} = \int d^3r \hat{\Psi}^\dagger(\mathbf{r}) \cos(kx) \cos(kz) \hat{\Psi}(\mathbf{r}) \)

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which measures the atomic density modulation on the checkerboard pattern $\cos(kx) \cos(kz)$.

The overall dispersive shift of the empty cavity resonance caused by the presence of the atoms is proportional to the bunching parameter $\hat{B} = \int d^3r \, \hat{\Psi}(r) \cos^2(kx) \hat{\Psi}(r)$.

After eliminating the steady-state cavity field of Eq. (S2) from Hamiltonian Eq. (S1), an effective Hamiltonian description is obtained (see main text, Eq. 1):

$\hat{H}_{\text{eff}} = \hat{H}_a + V \int d^3rd^3r' \, \hat{\Psi}(r) \hat{\Psi}(r') \cos(kx) \cos(kz) \cos(kx') \cos(kz') \hat{\Psi}(r) \hat{\Psi}(r')$.  \hspace{1cm} (S3)

The strength $V$ of this cavity-mediated atom-atom interaction is given by $V = \hbar \eta^2 \Delta_c \approx \hbar \eta^2 \Delta_c$, where the detuning of the pump laser from the dispersively shifted cavity resonance $\Delta_c = \Delta_c - U_0 B_0$ was taken to be large compared to the cavity half-linewidth $\kappa$. Here, $B_0 = \langle \hat{B} \rangle$ denotes the bunching parameter in the steady state. In a quantized picture, $NV/\hbar$ corresponds to the rate at which cavity photons are exchanged between atoms, as shown exemplarily in the zoom of Fig. 1A in the main text.

The effective Hamiltonian Eq. (S3) describes a closed system and neglects dynamical and quantum backaction effects originating from cavity decay and cavity input noise. This is justified on short timescales as long as $|\Delta_c| \gg \kappa \gg E_r / \hbar$ (35).

**Mean-field description in the steady state**

*Based on a mean-field description (36), we derive a numerical solution of the steady state of the system.*

A mean-field description of the system is obtained by formally replacing the operators $\hat{\Psi}$ and $\hat{a}$ in Eq. (S1) with the atomic mean-field $\sqrt{N}\psi_0$ and the coherent cavity amplitude $\alpha_0$, respectively. Their steady-state values are determined by the non-local Gross-Pitaevskii equation

$\mu_0 \psi_0(x, z) = \left( -\frac{\hbar^2}{2m} (\partial_x^2 + \partial_z^2) + V_p(z) + \hbar \eta(x, z) (\alpha_0 + \alpha_0^*) + \hbar U_0(x) |\alpha_0|^2 + g_{2D} |\psi_0|^2 \right) \psi_0(x, z)$. \hspace{1cm} (S4)
with \( \alpha_0 = \frac{\eta \Theta_0}{\Delta_c + i \kappa} \) and the chemical potential \( \mu_0 \). Here, we introduced the steady state order parameter \( \Theta_0 = N \langle \psi_0 | \cos(kx) \cos(kz) | \psi_0 \rangle \) and bunching parameter \( B_0 = N \langle \psi_0 | \cos^2(kx) | \psi_0 \rangle \), as well as the notations \( V_p(z) = V_p \cos^2(kz) \), \( \eta(x,z) = \eta \cos(kx) \cos(kz) \) and \( U_0(x) = U_0 \cos^2(kx) \). We reduced the description in Eq. (S4) to the pump and cavity directions, assuming a homogeneous condensate density along the weakly confined \( y \)-axis. The contact interaction strength is accordingly replaced by \( g_{2D} = \lambda^2 \bar{n}g \) with average 3D condensate density \( \bar{n} \) and the normalization condition \( \int_0^\lambda dx \int_0^\lambda dz |\psi_0|^2 = 1 \) (37).

For negative \( \tilde{\Delta}_c \), Eq. (S4) exhibits a dynamical instability above a critical transverse pump power \( P_{cr} \), driving the system from a normal phase into a supersolid phase with \( \lambda \)-periodic density modulation. In the normal phase, \( P < P_{cr} \), the condensate density is flat along the cavity axis, resulting in a vanishing order parameter, \( \Theta_0 = 0 \), and cavity field amplitude, \( \alpha_0 = 0 \). The mean-field solution \( \psi_0 \) is given by the lowest energy Bloch state in the shallow optical lattice potential of the transverse pump field. In the supersolid phase, \( P > P_{cr} \), the atomic cloud exhibits a \( \lambda \)-periodic density modulation both along the transverse and the cavity direction. Correspondingly, the order parameter takes a finite value, \( \Theta_0 \neq 0 \), and light scattering off the diagonal Bragg planes results in a coherent cavity field amplitude, \( \alpha_0 \neq 0 \). The mean-field solution \( \psi_0 \) is given by the minimal energy state in the two-dimensional lattice potential originating from interference between the transverse and cavity field. In the supersolid phase, the emergent checkerboard density modulation locks to one of two possible sublattices, which are spatially shifted by \( \lambda/2 \) and have opposite signs of \( \Theta_0 \) and \( \alpha_0 \). As both solutions share the same excitation spectrum, we assume \( \Theta_0 > 0 \) in the following.

We numerically find the ground state \( \psi_0 \) of the system by propagating Eq. (S4) in imaginary time, using a computational cell of size \( \lambda^2 \) with periodic boundary conditions. We include the recorded steady state mean intracavity photon number \( |\alpha_0|^2 \) and the experimentally calibrated lattice depth \( V_p \) in the calculations.
Deriving the collective excitation spectrum

We describe collective excitations on top of the mean-field solution. To this end, we calculate the Bogoliubov excitations in the steady-state lattice potential and identify a single excitation mode which is dominantly affected by the long-range interactions. We diagonalize the resulting truncated Hamiltonian and deduce the energy of collective excitations.

Following Ref. (36), we expand the atomic and cavity field operators around their mean-field solution \((\psi_0, \alpha_0)\) according to

\[
\hat{\Psi} = (\sqrt{N}\psi_0 + \delta\hat{\Psi})e^{-i\mu_0 t/\hbar} \\
\hat{a} = \alpha_0 + \delta\hat{a}.
\]

(S5)

Here, the linearized cavity field fluctuation operator

\[
\delta\hat{a} = \frac{\eta}{\Delta_c + i\kappa} \left[ \delta\hat{\Theta} + \Theta_0 U_0 \delta\hat{B} \right].
\]

(S6)

is given in terms of the operators \(\delta\hat{\Theta} = \sqrt{N} \int d^3r (\delta\hat{\Psi} + \delta\hat{\Psi}^\dagger) \cos(kx) \cos(kz)\psi_0\) and \(\delta\hat{B} = \sqrt{N} \int d^3r (\delta\hat{\Psi} + \delta\hat{\Psi}^\dagger) \cos^2(kx)\psi_0\), describing fluctuations of the order and bunching parameter around their steady-state values \(\Theta_0\) and \(B_0\), taking \(\psi_0\) to be real-valued.

After eliminating the cavity field fluctuation \(\delta\hat{a}\), assuming \(\Delta_c^2 \gg \kappa^2\), and keeping only linear terms in the equation of motion for \(\delta\hat{\Psi}\), we arrive at a quadratic Hamiltonian

\[
\hat{H}_{\text{exc}} = \hat{H}_0 + V \left[ \delta\hat{\Theta} + \frac{\Theta_0 U_0}{\Delta_c} \delta\hat{B} \right]^2
\]

(S7)

describing the dynamics of atomic fluctuations around the mean-field solution. The first term \(\hat{H}_0\) captures the dynamics in the static steady-state lattice potential including contact interactions, and is given by

\[
\hat{H}_0 = \int d^3r \delta\hat{\Psi} \left[ \frac{-\hbar^2}{2m}(\partial_x^2 + \partial_z^2) + V_p(z) + \hbar \eta(x,z)(\alpha_0 + \alpha_0^*) + \hbar U_0(x)|\alpha_0|^2 \right] \delta\hat{\Psi}
\]

\[
+ \frac{1}{2} g_{2D} |\psi_0|^2 \left( \delta\hat{\Psi}^2 + (\delta\hat{\Psi}^\dagger)^2 \right) + 2g_{2D} |\psi_0|^2 \delta\hat{\Psi}^\dagger \delta\hat{\Psi}.
\]

(S8)
The second term in Eq. (S7) describes how atomic density fluctuations are affected by the cavity-mediated long-range interaction.

In the normal phase, $\hat{H}_{\text{exc}}$ is directly obtained from Eq. (S3) by quadratic expansion in $\delta \hat{\Psi}$ around the steady state. In the supersolid phase, the dispersive cavity shift, which depends on the atomic density distribution, gives rise to an additional long-range interaction term with $\lambda/2$-periodicity along the cavity axis (34), whose strength is proportional to the order parameter $\Theta_0$. However, as the order parameter vanishes at the critical point, solely the term $\delta \hat{\Theta}^2$ in Eq. (S7) induces the normal to supersolid phase transition.

To find the energy of collective excitations described by $\hat{H}_{\text{exc}}$, we first calculate the elementary excitations of $\hat{H}_0$ using the Bogoliubov ansatz (38, 39)

$$\delta \hat{\Psi}(r) = \sum_j \left( u_j(r) \hat{c}_j + v_j^*(r) \hat{c}_j^\dagger \right),$$

with band index $j$, Bogoliubov modes $u_j(r)$ and $v_j(r)$, and corresponding mode operators $\hat{c}_j$, fulfilling bosonic commutation relations. In this basis the Hamiltonian $\hat{H}_{\text{exc}}$ reads

$$\hat{H}_{\text{exc}} = \sum_j \left( E_j \hat{c}_j^\dagger \hat{c}_j + NV \chi_j (\hat{c}_j + \hat{c}_j^\dagger)^2 \right),$$

with Bogoliubov energies $E_j$ and interaction matrix elements

$$\chi_j = \left\langle \psi_0 \left| \cos(kx) \cos(kz) + \frac{\Theta_0 U_0}{\Delta_e} \cos^2(kx) \right| u_j + v_j \right\rangle^2.$$

From a numerical calculation of the matrix elements $\chi_j$ we find in our parameter regime only a single Bogoliubov mode exhibiting a relevant matrix element $\chi_j$ for energies up to $15E_r$. All other matrix elements $\chi_j$ are suppressed by more than two orders of magnitude. In the normal phase the maximally coupled excited mode lies in the lowest energy band of the transverse lattice potential with quasi-momentum $(q_x, q_z) = (\pm \hbar k, \pm \hbar k)$. In the supersolid phase this state is folded into the third band at the center of the Brillouin zone as a result of the emerging $\lambda$-periodicity.
Figure S1: (A) Bare energies (blue) of the excited state with maximum matrix element $\chi$ in the normal and supersolid phase. Dashed and solid blue lines correspond to the case of vanishing and present contact atom-atom interactions. The red dashed-dotted line shows the soft mode energy in the presence of long-range and contact atom-atom interactions. (B) Maximum matrix element $\chi$ in the presence (solid) and absence (dashed) of contact atom-atom interactions.

In terms of the excited mode with dominant matrix element $\chi_j$, which we denote in the following by the index $j = 1$, the effective Hamiltonian reduces to

$$\hat{H}_{\text{exc}} = E_1 \hat{c}^\dagger_1 \hat{c}_1 + NV \chi (\hat{c}_1 + \hat{c}^\dagger_1)^2$$

(S12)

with $\chi = \chi_1$. In the absence of the transverse lattice potential and contact atom-atom interactions, $\hat{H}_{\text{exc}}$ reduces in the normal phase to the Hamiltonian given in the main text, Eq. 2.

We obtain the soft mode energy spectrum by diagonalizing Hamiltonian $\hat{H}_{\text{exc}}$ in terms of a second Bogoliubov transformation $\hat{b} = \mu \hat{c}_1 + \nu \hat{c}^\dagger_1$. Up to a constant term, this yields $\hat{H}_{\text{exc}} = E_s \hat{b}^\dagger \hat{b}$ with soft mode energy

$$E_s = E_1 \sqrt{1 + \frac{4NV\chi}{E_1}}.$$  

(S13)

For $V < 0$, $E_s$ softens towards the phase transition and vanishes at $V_{\text{cr}} = -E_1^{\text{cr}}/(4N\chi_{\text{cr}})$ with $(E_1^{\text{cr}}, \chi_{\text{cr}}) = (E_1, \chi)|_{P = P_{\text{cr}}}$. In the supersolid phase, the excitation energy rises again due to competition between the increasing Bogoliubov energy $E_1$ and the decreasing interaction energy which is proportional to the matrix element $\chi$ (see Fig. S1).
Results for two circularly polarized cavity modes $\hat{a}_\pm$ are obtained by replacing $V\chi$ in (S12) and (S13) by $\sum_{n=\pm} V_n \chi_n$, with $V_\pm$ and $\chi_\pm$ deduced accordingly from the cavity parameters $(\eta_\pm, U_0^{\pm})$. The calculated energy spectrum shown in Fig. 3 takes into account the experimentally calibrated depth of the transverse lattice potential and the measured steady-state intracavity photon number $|\alpha_0|^2$ in the presence of atoms. Systematic uncertainties of these quantities are estimated to be 10% and 25%, respectively (see shaded regions in Fig. 3).

**Probing the collective excitation spectrum**

By integrating the equation of motion of the atomic field, we predict the build-up of atomic excitations and of the cavity light field during probing.

To probe the excitation spectrum, we weakly drive the cavity field with amplitude $\eta_{pr}(t)$ and frequency $\omega_{pr}$. This is described by the driving Hamiltonian $-\hbar \eta_{pr}(t)(\hat{a}e^{i(\delta t + \phi)} + \hat{a}^\dagger e^{-i(\delta t + \phi)})$ with probe-pump detuning $\delta = \omega_{pr} - \omega_p$ and relative phase $\varphi$ between probe and pump beam. The resulting coherent intracavity probe field is given by

$$\alpha_{pr}(t) = -\frac{\eta_{pr}(t)e^{-i(\delta t + \phi)}}{\Delta_c + i\kappa}. \quad (S14)$$

Interference of the probe field with the transverse pump field and the steady-state cavity field $\alpha_0$ results in a modulated lattice potential. The corresponding perturbation of the atomic field is given by

$$\hat{H}_{pr} = \hbar \xi(t) \left[ \delta \hat{\Theta} + \frac{U_0 \Theta_0}{\Delta_c + i\kappa} \delta \hat{B} \right] \cos(\delta t + \varphi), \quad (S15)$$

with perturbation amplitude $\xi(t) = 2\eta\sqrt{n_{pr}(t)}$ and mean intracavity photon number $n_{pr}(t) = \frac{\eta_{pr}^2(t)}{\Delta_c^2 + \kappa^2}$. As $|\Delta_c| \gg \kappa$ in the experiment, we set the phase shift, originating from coupling into the cavity, to $\pi$. In terms of the mode operator $\hat{c}_1$, the perturbation reads

$$\hat{H}_{pr} = \hbar \xi(t) \sqrt{N\chi} (\hat{c}_1 + \hat{c}_1^\dagger) \cos(\delta t + \varphi). \quad (S16)$$
To quantify the response on this perturbation, we evolve the Hamiltonian $\hat{H} = \hat{H}_{\text{exc}} + \hat{H}_{\text{pr}}$ in time according to the Heisenberg equation

$$i\hbar \dot{\hat{c}}_1 = E_1 \hat{c}_1 + 2N V \chi (\hat{c}_1 + \hat{c}_1^\dagger) + \hbar \xi(t) \sqrt{N \chi} \cos(\delta t + \varphi) - i\hbar \gamma \hat{c}_1. \quad (S17)$$

Motivated by our experimental observations, we phenomenologically introduced a damping term with damping constant $\gamma$ into the time evolution of $\hat{c}_1$. This accounts for possible damping mechanisms like s-wave scattering with other momentum modes, trap loss or finite-size dephasing. The general solution of Eq. (S17) reads

$$\hat{c}_1(t) = 2 \eta \sqrt{N \chi n_{\text{pr},0}} \left( \frac{E_1}{E_s} \text{Im}(Y(t)) + i \text{Re}(Y(t)) \right). \quad (S18)$$

Here,

$$Y(t) = e^{(i\omega_s - \gamma)t} \int_0^t dt' e^{-(i\omega_s - \gamma)t'} \cos(\delta t' + \varphi) \Pi(t'), \quad (S19)$$

with $\omega_s = E_s/\hbar$ and $n_{\text{pr}}(t) = n_{\text{pr},0} \Pi(t)^2$. Here, the envelope function $\Pi(t)$ of the probe pulse with duration $\tau$ and the maximum probe photon number $n_{\text{pr},0}$ were introduced. The population of the excited momentum mode and the mean intracavity photon number are directly obtained from Eq. (S18)

$$N_1(t) = \langle \hat{c}_1^\dagger \hat{c}_1 \rangle = 4 \eta^2 n_{\text{pr},0} N \chi \left[ \left( \frac{E_1}{E_s} \right)^2 \text{Im}(Y(t))^2 + \text{Re}(Y(t))^2 \right]$$

$$n_{\text{ph}}(t) = \left| \alpha_0 - \frac{4\eta^2 \sqrt{n_{\text{pr},0} N \chi}}{\Delta_e + i\kappa} \left( \frac{E_1}{E_s} \right) \text{Im}(Y(t)) + \sqrt{n_{\text{pr}}(t)} e^{-i(\delta t + \varphi)} \right|^2. \quad (S20)$$

The second term in $n_{\text{ph}}(t)$ describes the pump field which was Bragg scattered off the excited density modulation into the cavity mode. From $N_1$, the population $N_e$ in the momentum state $|e\rangle$ is obtained according to $N_e = \zeta N_1$, where $\zeta$ denotes the absolute square of the Fourier amplitude of the excited Bogoliubov mode $u_1 + v_1$ at momenta ($\pm \hbar k, \pm \hbar k$).

The curves shown in Fig. 2C and D of the main text are obtained from Eqs. (S20), where the phase $\varphi$ was adjusted to fit the data in Fig. 2C. The resonance curves, displayed in Fig. 2B and E
of the main text, correspond to phase-averaged values \( \langle N_e(\tau) \rangle_{\varphi \in [0, 2\pi]} \) and \( \langle \int_0^\tau dt \, n_{ph}(t) / \tau \rangle_{\varphi \in [0, 2\pi]} \) where the resonance frequency \( \omega_s \) and the amplitude \( \eta^2 N \chi \) were adjusted independently to fit the data. Due to the detection background an offset was added to \( N_e(\tau) \). The standard deviation of fluctuations associated with the uncontrolled relative phase \( \varphi \) is displayed by the shadings in Fig. 2B and E of the main text.

**Response in atomic density and momentum state population**

Bragg spectroscopy, as described by the density perturbation \( \hat{H}_{pr} \), induces a response in the atomic density and in the population of the excited state. Normalized to the integrated amplitude of the perturbation, this density response is a measure for the susceptibility of the system on an external density perturbation and provides at zero temperature a direct link to the static structure factor \((38,40)\). We use the recorded modulation \( \langle \delta \hat{a} \rangle \) of the intracavity field to quantify the density response \( \langle \delta \hat{\rho} \rangle \) of the system.

Consider the density operator \( \hat{\rho}(r) = \hat{\Psi}(r)^\dagger \hat{\Psi}(r) \) and its linear expansion \( \hat{\rho} = \rho_0 + \delta \hat{\rho} = N|\psi_0|^2 + \sqrt{N}\psi_0 (\delta \hat{\Psi}^\dagger + \delta \hat{\Psi}) \) around the equilibrium value \( \rho_0 = \langle \hat{\rho} \rangle \). In terms of the density fluctuation operator in Fourier space, defined as \( \delta \hat{\rho}_k = \int d^3 r \, e^{-ikr} \delta \hat{\rho}(r) \), the fluctuations of the order and bunching parameter \( \delta \hat{\Theta} \) and \( \delta \hat{B} \) read

\[
\delta \hat{\Theta} = \sum_{k \in (\pm k, \pm k)} \delta \hat{\rho}_k / 4 \quad \text{and} \quad \delta \hat{B} = \sum_{k \in (\pm 2k, 0)} \delta \hat{\rho}_k / 4 . \tag{S21}
\]

Fluctuations of the cavity field \( \delta \hat{a} \), see Eq. (S6), are thus given by

\[
\delta \hat{a} = \frac{\eta}{4(\Delta_c + i\kappa)} \left[ \sum_{k \in (\pm k, \pm k)} \delta \hat{\rho}_k + \frac{\Theta_0 U_0}{\Delta_c + i\kappa} \sum_{k \in (\pm 2k, 0)} \delta \hat{\rho}_k \right] , \tag{S22}
\]

and provide a measure of the induced atomic density modulation. We define the corresponding (quadratic) density response function \( \mathcal{R}_\rho \) as

\[
\mathcal{R}_\rho = \frac{1}{P \int_0^\tau dt \, n_{pr}(t)} \int_{-\infty}^\infty d\delta \left\langle \int_0^\tau dt \frac{|\langle \delta \hat{a} \rangle|^2}{\eta/(4(\Delta_c + i\kappa))^2} \right\rangle_{\varphi \in [0, 2\pi]} , \tag{S23}
\]
where again an average over the relative phase $\varphi$ is performed.

Experimentally, we extract $|\langle \delta \hat{a} \rangle|^2$ from the resonance fits based on Eq. (S20) (see Fig. 2E in the main text), with experimentally calibrated $|\alpha_0|^2$ and $n_{pr}$. In order to probe the linear response of the system, the probe power was accordingly lowered when approaching the critical point. The lowest perturbation applied in the normal phase corresponds to 30(8) intracavity probe photons in total.

Similarly, the response function $\mathcal{R}_N$ associated with the detected number of atoms with momenta $(\pm \hbar k, \pm \hbar k)$ is defined as

$$\mathcal{R}_N = \frac{1}{P} \int_0^\infty dt \int_{-\infty}^{\infty} d\delta \langle N_e(\tau) \rangle_{\varphi \in [0,2\pi]},$$

which is proportional to the area below the fitted resonances of $N_e(\tau)$, see Fig. 2B in the main text.

The dominant scaling factor of $\mathcal{R}_\rho$ when approaching the critical point is found from Eq. (S20) to be $(E_1/E_s)^2$. As $\mathcal{R}_\rho$ measures the quadratic density response, this is in agreement with Feynman’s relation, given in the main text. Since $N_e(\tau)$ is sensitive to both quadratures of $\hat{c}_1$ (see Eq. (S20)), the response function $\mathcal{R}_N$ scales differently and exhibits larger variances in the relative phase $\varphi$.

References


References


25. See supplementary materials on *Science* Online.


References


25. See supplementary materials on *Science* Online.


