Supplementary Materials for

Regular Patterns in Frictional Resistance of Ice-Stream Beds Seen by Surface Data Inversion

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This PDF file includes:

Methods
Fig. S1 to S3
Table S1
References (25–30)
Methods

1 Inversion method

1.1 Method formulation

The inverse technique is the widely used control method (6), where the discrepancy between observed and simulated ice-stream surface velocities, quantified by a cost function $J$, is minimized in a least squares sense, employing Lagrange multipliers to impose the forward ice-flow model as a constraint on the minimization. The cost function is

$$J = \frac{1}{2} \iint_{A_s} [(u_s - u^*)^2 + (v_s - v^*)^2] + \iint_V \Lambda \cdot F,$$

where $u_s$, $v_s$, $u^*$, $v^*$ are the computed and observed ice surface horizontal velocity components respectively, $A_s$ and $V$ are the ice-stream surface area and volume, $\Lambda$ are the Lagrange multipliers and $F$ is the ice-flow forward model. The latter comprises the Stokes equations and ice incompressibility equation

$$F = \begin{bmatrix} \nabla \cdot \sigma - \rho g \bar{g} \\ \nabla \cdot u \end{bmatrix} = 0,$$

where $\nabla$ is the divergence operator, $\sigma = \sigma' - Ip$ is the Cauchy stress-tensor, $\sigma'$ is the deviatoric stress-tensor, $p$ is ice pressure, $I$ is the identity tensor; $\rho_i$ is ice density, $\bar{g}$ is the acceleration due to gravity, and $u = \{u, v, w\}$ is the ice velocity. The deviatoric stress-tensor $\sigma'$ is related to the strain-rate tensor $\dot{e} = \frac{1}{2} \left( \nabla u + (\nabla u)^T \right)$ by the constitutive relationship

$$\sigma' = 2 \nu \dot{e},$$

where $\nu$ is the stress-dependent ice viscosity modeled here by Glen’s flow law

$$\nu = \frac{1}{2} A r \pi \dot{e}^{\frac{1-n}{n}}.$$
Here $Ar$ is the temperature-dependent rate factor, $n$ is the Glen’s flow law exponent (taken to be 3) and $\dot{e}$ is the second invariant of the strain-rate tensor.

The boundary conditions of the forward model $F$ are

\[
\begin{align*}
\sigma \cdot \vec{n} \big|_{z=s} &= 0, \\
\tau_b \big|_{z=b} &= \beta \vec{u}, \\
\vec{u} \cdot \vec{n} \big|_{z=b} &= 0, \\
\sigma \cdot \vec{n} \big|_{\Gamma} &= -\vec{n} p_i,
\end{align*}
\]  

(5a) \hspace{1cm} (5b) \hspace{1cm} (5c) \hspace{1cm} (5d)

where $s$ and $b$ are elevations of the top and bottom ice surfaces, $\vec{n}$ is an outward pointing normal to these surfaces, $\tau_b$ is the basal traction, $\beta$ is spatially variable basal traction parameter; $\Gamma$ is the vertical surface of the domain boundaries shown by blue lines in Fig. 1, and $p_i$ ice overburden.

The inversion is used to estimate the basal traction parameter $\beta$. To prevent data over-fitting, i.e. creation of artifacts in the inverted basal shear caused by errors in the observed surface velocity data, a stopping criterion based on the discrepancy principle (25) was implemented as a point-wise constraint; an additional point-wise constraint on the smoothness of the ice strain-rates was introduced in order to prevent unphysical small-scale variations in the basal shear stress-strain rates were not allowed to exceed $10^{-8}$ s$^{-1}$. All computations in this section are performed using a finite-element solver COMSOL$^\text{TM}$. The mesh resolution varies from $\frac{1}{3}$ to $\frac{1}{5}$ of the local ice thickness, with finer resolution in areas closer to the grounding line.

1.2 Robustness of inverted spatial patterns

The robustness of the inverted spatial patterns was investigated using two different approaches. In the first approach, we applied Tikhonov regularization, and modified the cost function, $J$, by adding a regularization term $\alpha K \int_{A_b} \left| \vec{\nabla} \beta \right|^2$, where $A_b$ is the bed area of the ice streams, $\alpha$ is the regularization parameter, and $K$ is a normalization coefficient, such that for $\alpha=1$, the first term on the right-hand side of expression (1) and the regularization term are of the same order of magnitude. The regularization term enforces smoothness of the inverted basal roughness parameter $\beta$. Rather than selecting a particular value of $\alpha$, e.g. by using the $L$-curve method (26), so as to avoid the possibility of non-convergence (27) we consider a suite of regularization parameters $\alpha$ to demonstrate that the spatial features are robust regardless of the degree of smoothing. These tests were performed on subdomains of Thwaites and Pine Island glaciers near their grounding lines (subdomain of the PIG is shown in Fig. 4 of the main text). Figure 1M shows inverted basal shear $\tau_b$ for different values of the regularization parameter. While details of the basal distribution differ for different levels of smoothness, the distinct rib-like patterns are clearly present in all realizations. Further reduction of $\alpha$ yields the spatial distributions shown in Figs. 2 and 4 of the main text. We have also performed inversions using different initial guesses of the basal traction - uniform, uniform with 20% amplitude random noise, and setting the basal traction to the driving stress. In all cases, the inverted basal shear distributions look similar to each other.

In the second approach, we examined the ability of the inverse method to retrieve spatial distribution of the basal traction using synthetic configurations. We employed a two-dimensional ($x$-$z$ plane) flow model simulating ice flow over two adjacent high basal traction patches to create
synthetic flow fields and ice geometries to which we applied the inverse method, retrieving the spatial distribution of basal shear and comparing this with the value prescribed in the forward model. Using this approach, we performed a parametric sweep on the spacing between these two patches, which were set to have width one ice thickness, computing steady ice-surface elevation and ice surface velocities for patch spacings of 1-10 ice thicknesses. The model simulates the plane-flow of an ice stream 100 km long and 1500 m thick flowing on inclined plain with angle 0.2°. The patches with the high basal traction are equidistant from the center of the domain, with magnitudes of basal traction 400 times larger than the ambient values.

A particular objective was investigating the minimum spacing between the patches with high basal traction that can be robustly retrieved by the inverse method. To this end, the computed surface elevation, ice surface velocity and bed elevation were used as observational inputs for the inverse model as described above. Furthermore, to simulate possible observational errors and inconsistency between the observations (e.g. compilation of ice elevations observed at different times), random spatially uncorrelated distributed noise with 1 km wavelength (the grid spacing of the available data sets) was added to the bed elevation (50 m amplitude), surface elevation (10 m amplitude), and ice surface velocity (30 m yr\(^{-1}\) amplitude). These magnitudes reflect observational errors of the corresponding fields in our inversions of Antarctic data. The inversions of these synthetic observations were carried out using the inversion method described above. Fig. 2M(a) shows an example of an inversion for a rib spacing of two ice-thicknesses between the high basal-traction patches. While the amplitudes of the inverted basal traction (solid line) differ from the prescribed ones, the spatial distributions are virtually identical. Figs. 2M(b) and 2M(c) summarize the results of inversions for patch spacings. In all inversion experiments the retrieved spacing is robust. It is notable that smoothing (Fig. 2M(b)) creates more uniform peak values, at expense of delocalization.

On the basis of the analysis performed, we conclude that the rib-like patterns in the basal shear of Pine Island and Thwaites glaciers are robust features, with rib-spacing insensitive to parameters of the inverse method and observational errors.

2 Basal process modeling

2.1 Model description

To explore developments of possible instabilities at the base of the ice streams in the presence of subglacial water, we use a coupled model of an ice stream with deformable till underlying it, and subglacial water flow at the ice/till interface. The ice-stream flow model is the same as above, (eqns (2)).

Sediment thickness \(D\) evolves according to the Exner equation (21)

\[
\frac{\partial D}{\partial t} + \nabla \cdot \mathbf{q} = 0,
\]  

(6)

where the sediment flux \(\mathbf{q}\) is function of the basal stress and effective pressure, \(p_e = p_i - p_w\) with \(p_w\) being subglacial water pressure.

The basal velocity is given by a combination of deformation (the rate coefficient \(A_d\)) and sliding
over the till (rate coefficient $A_s$)

$$u = \frac{(A_d + A_s) |\tau_b|^{\ell-1}}{p_e^\ell} \tau_b,$$

(7)

where $\ell$ is the sliding index, and the till flux due to internal deformation is given by

$$q = \frac{A_d |\tau_b|^{\ell-1}}{(\rho_s - \rho_w) (1 - \phi) |g_e| (\ell + 1) p_e^{\ell+1} \tau_b},$$

(8)

where $\phi$ is the sediment porosity, $\rho_s$ the sediment density and $\rho_w$ the density of water. These equations are found in e.g. (20). The rate coefficients are unconstrained by data and we set them by choosing a basal ice velocity as part of a parameter exploration and distributing this between ice-till sliding and till internal deformation, again as part of a parameter exploration. The value of the index $\ell$ is poorly known in geophysical situations and we vary it between low quasi-viscous values ($\ell = 3$) to quasi-plastic values ($\ell = 10$). In practice this does not make much quantitative difference to predictions of wavelength. In some simulations we also consider the effect a distributed drainage system on the effective pressure distribution; in other words we no longer assume hydraulic gradients to be negligible in the region of interest. We use a heuristic function $\kappa$ to describe effective-pressure-dependent transmissibility at the ice bed interface

$$q_w = -\kappa(p_e) \nabla \psi$$

(9)

where $k$ is the aquifer permeability. We model $\kappa$ as

$$\kappa(p_e) = \kappa_c/p_e^\lambda$$

(10)

where $\kappa_c$ is a constant. Water storage is also set to be effective pressure dependent

$$S(p_e) = S_i/p_e^\xi,$$

$$\partial_t S + \nabla \cdot q_w = 0.$$  

(11a)  

(11b)

More details are given by (20). The quantitative aspects of this model are poorly constrained by observations. Ignoring melting, the kinematic basal boundary condition is given by

$$\partial_t b + u \cdot \nabla b = w + \partial_t D + \partial_t S.$$  

(12)

For the prognostic model, the approach is based on the linearized approach (20, 28) and we refer the reader to these papers for details. Ref. (20) considers coupled ice/deforming bed flow at wavelengths less than the thickness of ice, using an analytic model of ice flow. We refer to equations from this paper by H98:xx, where xx is the equation number.

We combine the basal boundary conditions for till deformation (20) with a more appropriate mechanical model (28). This model, which is extensively tested against analytical solutions (28), treats the mechanics of ice flow over the range of wavelengths appropriate to traction ribs, and the two approaches (20, 28) are combined in a straightforward manner. Specifically, in addition to the momentum balance equations (H04:1,H04:xx) refers to equation xx in Ref. (28) which in linearized form is given by(H04:A1); we use the mass conservation condition

$$\partial_t H + \nabla \cdot (\bar{u} H) = 0,$$

(13)
(linearized form H04:35) to determine the evolution of ice thickness where \( t \) is time, \( \mathbf{u} \) is the horizontal velocity vector, the bar refers to its vertical average at any point. The basal boundary conditions are as given in (20), equations (H98:15-H98:16) for till internal deformation, with linearized forms given by (H98:8,H98:9,H98:11,H98:12). These boundary conditions are used with the Stokes equations linearized boundary tractions (H04:A9), with the final equation in that set replaced by

\[
\hat{w}^{(1)}_b = \frac{d\hat{D}^{(1)}}{dt} + \frac{d\hat{S}^{(1)}}{dt} - i\hat{b}^{(1)}_b \mathbf{u}^{(0)}_b \cdot \mathbf{k},
\]

where the circumflex and superscript \((1)\) represent the first-order Fourier components (see (28) for details).

The prescription of sediment flux is poorly constrained by data, and our motivations for using it are its physical reasonableness and its ability to explain some classes of sub-glacial landforms as a pattern-forming instability. Likewise, the hydrology model is speculative; it depends upon the idea that the conductivity and the storage both increase as effective pressure decrease (29).

The field variables are linearized with a small parameter \( \mu \) about the base case solution of steady uniform flow down the infinite plane e.g. \( H = H^{(0)} + \mu H^{(1)}(\mathbf{r},t) \), etc.. These are used to derive a set of zeroth- and first-order equations expressing conservation of mass and momentum. A Fourier transform in the horizontal plane is then applied to the first-order equations, and the first-order fields can be expressed as plane waves \( H^{(1)} = \mathfrak{R} \left\{ \hat{H}^{(1)} \exp(\omega t - i\mathbf{k} \cdot \mathbf{r}) \right\} \), etc., where \( \omega \) is the eigenvalue or growth rate, the wavenumbers are given by \( \mathbf{k} = (k_x,k_y) \) and the caret indicates the Fourier coefficient of the transform over the \( \mathbf{r} \)-plane only, and similarly for the till thickness \( \hat{D} \) and water pressure \( \hat{p}_w \) Fourier components. Details are given in (28). We follow the customary ansatz of assuming patterns are formed at the wavenumber \( \mathbf{k} \) where the growth rate \( \omega \) is greatest. At each wave-number, three eigenvalues and three corresponding eigenvectors (with three components corresponding to ice thickness, till thickness and water pressure) proportion are computed.

### 2.2 Modeling results of structure formation

We explore parameter space to determine the spatial scales at which structure forms by searching for wave-numbers where instabilities grow, and also obtain the associated timescales. Results are shown as a ‘dispersion diagram’ (Fig. 3M) - a plot of growth-rate of a perturbation against wavelength. The color coding indicates the proportion of ice, till or water in the instability. This is shown because, in principle, the instability might act in the ice-thickness, the till thickness or the water pressure, or more usually, in a combination of these variables. The proportion is given by absolute value of the associated component of the eigenvector associated with the unstable mode (in practice, only one of the three modes is unstable). Note that if the ice surface is undisturbed, there will be an instability in the ice-thickness, while if the ice surface moves vertically in accordance with the vertical motion provided by till and/or water, there is no instability in the ice thickness. Initially structure is formed at the wavelength where the instability is growing fastest, and the hypothesis is that this is reflected in the ultimate structure.

Five plane flow \((k_y = 0)\) examples are presented, showing instabilities at a certain wavelength, with associated time-constants. The parameters that are varied are presented in Table 1, along
with results from the stability analysis. The constant conditions are an accumulation rate of 0.1 m a\(^{-1}\), an ice thickness of 2000 m, an internal temperature distribution given by the Robin solution using the same accumulation rate, a surface temperature of -30\(^\circ\)C and assuming that there is sufficient heat supply to maintain the base at melting point. The ice-rate factor at melting point \(A_{r0} = 1.5 \cdot 10^{-16} \text{ Pa}^{-3}\), and the temperature dependence of the rate factor is specified by 

\[
R = \frac{Ar}{A_{r0}} \quad \text{a dual-Arhenius relationship}
\]

\[
R(\theta, H) = c_1 \exp\left(-\frac{E_1}{G_c \theta}\right) + c_2 \exp\left(-\frac{E_2}{G_c \theta}\right)
\]

where \((c_1, c_2) = (3.7 \times 10^6, 5.4 \times 10^{26})\), \((E_1, E_2) = (60, 140)\text{kJ mol}^{-1}\) and \(G_c = 8.314 \text{ Jmol}^{-1}\) is the universal gas constant. Here, \(\theta\) is the homologous temperature. The hydraulic diffusivity was set to \(10^8 \text{ m}^2\text{a}^{-1}\), corresponding to a hydraulic propagation length of 10 km per year, a total sediment thickness of 10 m, with the conductivity index \(\lambda\) set to 3. All these values affect the quantitative results presented below; our intention is to demonstrate that the same governing equations that predict the formation of geomorphological features (ribbed moraine) can also produce structure at the wavelength observed, as a consequence of the unstable operation of the basal till/water system. The parameters varied were the base-case shear stress \(\tau_b\), the base-case effective pressure \(p_e\), the sliding index \(\ell\), the basal ice velocity \(u_b\) and the proportion of till internal deformation \(\theta\) contributing to the basal motion. The first case (the circles) corresponds to ribbed moraine formation discussed in (22), while the remainder exemplify the wide range of spatial scales that can be generated by the model under plausible parameter variations, as well as showing that the instability can be expressed mainly in the water (Case 3) or in mixtures of till and water. When the amount of slip is high, the features take a longer time to grow. Given the range of parameters, the model is not really testable, but it shows that a simple model can generate the phenomena discussed in the main text. Case 1 corresponds to ribbed moraine (22); Case 2 has wavelength comparable with the effective pressure stripes of (30); Cases 3-5 to features with wavelengths comparable to the observed traction ribs. However, Case 5, with wavelength 16km, has low ice-bed coupling, and has an \(e\)-folding growth time of 2000 years, which can be regarded as stable.

It is likely that the friction that determines the large scale flow of an ice-stream is determined on length scales greater than those of the ribs, and that in consequence a spatially averaged bulk value of the friction is all that is needed to understand ice-stream mechanics. However, since a pattern-forming instability generates traction ribs whose characteristics such as amplitude and wavelength may depend in a complex way on the effective pressure and basal shear stress, a bulk effective pressure/sliding coefficient relationship averaged over several ribs is likely to be more complex than present models that average over small-scale roughness; the same applies to effective pressure/conductivity relationships.
Table 1: Model parameters and results: inputs are basal shear stress $\tau_b$, effective pressure $p_e$, sliding index $\ell$, ice-stream basal velocity $u_b$, proportion of basal motion due to till interal deformation $\theta$, water layer thickness $S$. Predictions are wavelength $2\pi k^{-1}$, growth rate $\omega$, and expression of instability in water layer thickness and till layer thickness.

<table>
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<th>Case</th>
<th>$\tau_b$ (kPa)</th>
<th>$p_e$ (kPa)</th>
<th>$\ell$</th>
<th>$u_b$ (m a$^{-1}$)</th>
<th>$\theta$</th>
<th>$S$ (m)</th>
<th>$2\pi k^{-1}$ (km)</th>
<th>$\omega$ (a$^{-1}$)</th>
<th>Water</th>
<th>Till</th>
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References


Figure 1M: Dependence of basal shear $\tau_b$ on the Tikhonov regularization parameter $\alpha$: (a)-(c) Thwaites Glacier; (d)-(f) Pine Island Glacier.
Figure 2M: Robustness of the rib retrieval: (a) example of a 2d synthetic test with a two ice-thickness gap between two ribs with strong basal shear: the colored shape shows the computed steady glacier geometry, color is computed ice speed (m yr$^{-1}$), black dashed line is the distribution of prescribed basal slipperiness coefficient $\beta$, solid grey line retrieved $\beta$; (b)-(c) inverted basal roughness $\beta$ for different distances between the ribs: (b) Tikhonov regularisation coefficient $\alpha = \{0.5, 0.1, 0.02\}$ (c) $\alpha = 0$. White lines indicate locations of the patches with high basal tractions. Notice that regularization (smoothing) creates more uniform peak values, at the expense of delocalization.
Figure 3M: Instability growth-rate $\omega$ against wavelength $2\pi k^{-1}$. The color coding indicates the proportion of ice, till or water in the instability.
References and Notes


