Supplementary Materials for

A mixture of Bose and Fermi superfluids

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Supplementary material to
A Mixture of Bose and Fermi Superfluids

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Feshbach Resonances

The Bose-Fermi mixture is composed of a $^7$Li cloud prepared in the $|2\rangle_b$ state, which connects to the $|F = 1, m_f = 0\rangle$ state at low field, together with a $^6$Li gas in the two lowest energy states $|1\rangle_f$ and $|2\rangle_f$ connecting to $|F = 1/2, m_f = 1/2\rangle$ and $|F = 1/2, m_f = -1/2\rangle$ respectively.

In Fig. S1 we present the relevant s-wave scattering lengths characterizing the $^7$Li-$^7$Li, $^6$Li-$^6$Li and $^6$Li-$^7$Li interactions in the 700 G-1000 G magnetic field region of interest. $^7$Li, $|2\rangle_b$ exhibits two Feshbach resonances located at 845.5 G and 894 G. For fermionic $^6$Li, the two spin-states $|1\rangle_f$, $|2\rangle_f$ exhibit one very broad s-wave resonance at 832.18 G. Note the 1/100 vertical scale for $^6$Li in Fig. S1. The scattering lengths are taken from (11,31) in units of Bohr radius $a_0$ as a function of magnetic field $B$ in gauss:

$$a_f(B) = -1582 \left( 1 - \frac{-262.3}{B - 832.18} \right)$$

$$a_b(B) = -18.24 \left( 1 - \frac{-237.8}{B - 893.95} \right) \left( 1 - \frac{4.518}{B - 845.54} \right)$$

For the inter-isotope interaction, coupled-channel calculations by S. Kokkelmans provide a scattering length $a_{bf} = 40.8 \ a_0$ independent of the magnetic field in this region.

![Figure S1: Magnetic field dependence of the different scattering lengths $a_b$ (blue), $a_f$ (red), and $a_{bf}$ (dashed gray). $a_{bf} = 40.8 \ a_0$ is independent of $B$. Note the 1/100 vertical scale for $^6$Li.](image-url)
Experimental set-up, mixture preparation

The apparatus and early stages of our experiment have been described in (32). Initially $^7$Li (resp. $^6$Li) atoms are cooled to 40 $\mu$K in a Ioffe-Pritchard trap in the $|F = 2, m_f = 2\rangle$ (resp. $|F = 3/2, m_f = 3/2\rangle$) states at a bias field of 12.9 G. The trapping potential for the mixture is a hybrid trap composed of an optical dipole trap (wavelength 1.07 $\mu$m) with waist 27 $\mu$m superimposed with a magnetic curvature in which the bias magnetic field remains freely adjustable. About $3 \times 10^5$ $^7$Li and $2 \times 10^6$ $^6$Li atoms are transferred in a 300 $\mu$K deep optical dipole trap. They are then transferred to their absolute ground state $|F = 1, m_f = 1\rangle$ and $|F = 1/2, m_f = 1/2\rangle$ by a rapid adiabatic passage (RAP) using two 50 ms radio-frequency (RF) pulses and a sweep of the magnetic bias down to 4.3 G. These states connect at high magnetic field respectively to $|1_b\rangle$ and $|1_f\rangle$. We revert the magnetic curvature in order to provide an axial confining potential, the ground states being high-field-seeking states. The bias field is ramped in 100 ms to 656 G where we transfer $^7$Li to the state $|2_f\rangle$ by a RAP done by an RF pulse with a frequency sweep from 170.9 MHz to 170.7 MHz in 5 ms. The field is ramped in 100 ms to 835 G where a mixture of $^6$Li in its two lowest energy states $|1_f\rangle$ and $|2_f\rangle$ is prepared with an RF sweep between 76.35 MHz and 76.25 MHz. The duration of this sweep varies the Landau-Zener efficiency of the transfer offering control of the spin polarization of the $^6$Li mixture. Initial conditions for evaporation at this field are $1.5 \times 10^5$ $^7$Li and $1.5 \times 10^6$ $^6$Li at 30 $\mu$K in a 300 $\mu$K deep trap. The evaporation of the mixture is done near unitarity for the fermions providing high collision rate. In 3 s the laser power is reduced by a factor 100 and $^7$Li is sympathetically cooled by $^6$Li with high efficiency; the phase-space density increases to BEC by a factor $\sim 2 \times 10^4$ for a factor of ten loss in $^7$Li atoms. To confirm this sympathetic cooling scheme we have also performed the evaporation at 850 G where the $^7$Li scattering length vanishes, demonstrating that $^7$Li can be cooled down solely by thermalisation with $^6$Li. At the end of evaporation, we typically wait 700 ms at constant dipole trap power to ensure thermal equilibrium between both species.

The trapping potential is cylindrically symmetric, with axial (transverse) frequency $\omega_z$ ($\omega_\rho$). The BEC phase transition is observed at a temperature of 700 nK. Our studies are performed in a shallow trap with frequencies:

- $\omega_{\rho,b} = 2\pi \times 550(20)$ Hz, $\omega_{\rho,f} = 2\pi \times 595(20)$ Hz
- $\omega_{z,b} = 2\pi \times 15.27$ Hz, $\omega_{z,f} = 2\pi \times 16.8$ Hz.

These frequencies are measured by single species center-of-mass oscillations at a field of 832G.

Typical atoms numbers are $N_b = 4 \times 10^4$ $^7$Li atoms and $N_f = 3.5 \times 10^5$ $^6$Li in a spin-balanced mixture. The critical temperature for $^7$Li Bose-Einstein condensation is $T_{c,b} = \frac{\hbar \omega_b}{k_B} \left(N_b/\zeta(3)\right)^{1/3} = 260$ nK and the Fermi temperature for $^6$Li $T_F = \frac{\hbar \omega_\rho}{k_B} (3N_f)^{1/3} = 880$ nK. To our experimental precision, the condensed fraction $\frac{N_\rho}{N}$ is higher than 0.8, implying $\frac{T_F}{T_c,b} \lesssim 0.5$. With $T_f \lesssim T_b$, we have $\frac{T_F}{T_c,f} \lesssim 0.15 = 0.8 T_{c,f}$. This temperature upper bound indicates fermionic superfluidity, in agreement with the direct observation of the superfluid core in the spin-imbalanced gas shown in Fig. 1 in the main text and the extremely low damping observed for small relative oscillations between both isotopes.

The large imbalance in isotope population $N_f/N_b \sim 10$ results from our cooling strategy. At the cost of a small loss in $^6$Li numbers, we can also get samples containing $N_b \sim N_{f,\uparrow} \sim N_{f,\downarrow} \sim 10^5$.

To excite the dipole mode of the two superfluids, we take advantage of the fact that the axial position of the waist of the dipole trap laser beam is slightly off-centered with respect to the minimum of the axial magnetic confinement. In order to displace the center of the atomic clouds, we slowly increase the laser power of the dipole trap by a variable factor (between 1.1 and 2). This results in axial displacement and radial compression of both clouds. The intensity ramp is done in $t_{\text{up}} = 150$ ms, i.e slow compared to the
trap periods. We then return the laser power to its initial value in $t_{\text{down}} = 20$ ms, fast compared to the axial trap period but slow compared to the radial period, avoiding excitation of radial collective modes. The center of mass positions of both clouds are measured by recording in situ images at variable delays after the axial excitation, up to 4 seconds. Examples of center-of-mass oscillations over a time span of more than 3.5 s are shown in Fig S2.

![Figure S2: Examples of center-of-mass oscillations. $^7$Li bosons alone at 832 G (a), $^7$Li bosons in the presence of $^6$Li fermions at 832 G (b). 4 second time span for the evolution of $^7$Li bosons (c) mixed with $^6$Li fermions (d) at 835 G. The coherent energy exchange between $^7$Li and $^6$Li superfluids with no detectable damping is clearly visible.](image)

**Critical velocity measurement**

When we increase the initial amplitude $d_0$ of the oscillations above $\approx 100 \mu m$, we observe first strong damping of the $^7$Li BEC oscillations inside the Fermi cloud followed by long-lived oscillations at a lower amplitude, as shown in Fig. S3. To verify that this damping is not due to trap anharmonicity for large displacements, we measured oscillations of the BEC in the absence of fermions. For a displacement of $d = 120 \mu m$, which corresponds to a velocity of $v \approx 0.45v_F$ in the presence of the Fermi cloud, we found a characteristic damping rate of $\gamma = 0.05$ s$^{-1}$. For a much larger initial displacement $d = 275 \mu m$ ($v \approx v_F$) we observe an influence of trap anharmonicity with an effective damping rate $\gamma = 0.26$ s$^{-1}$. Both of these rates are much smaller than the measured rates in presence of the Fermi cloud for velocities above 0.4 $v_F$ as shown in Fig. 2(c) in main text.

The observed behavior is compatible with a critical velocity $v_c$ for relative motion, resulting in damping for velocities above $v_c$ at early times and then undamped oscillations when the velocity is smaller than
We fit our data with Eq. (7) from main text and an amplitude $d = d_0 \exp(-\gamma t) + d'$ where $d'$ is the amplitude for the long-lived final oscillations. $\gamma$ is then a damping rate extracted from each data set. Its variation against maximal relative velocity between the two clouds is shown in Fig. 2(c) of main text. To extract a critical velocity we use a simple model:

$$\gamma(v) = \Theta(v - v_c) \ A \ (v - v_c)/v_F^\alpha$$

(S3)

where $\Theta(x)$ is the Heaviside function, $A$ and $\alpha$ are free parameters. By fitting Eq. (S3), we obtain a critical velocity $v_c = 0.42^{+0.05}_{-0.01} v_F$, an exponent $\alpha = 0.95^{+0.8}_{-0.3}$, close to 1, and $A = 17(9) s^{-1}$. This function is plotted in solid blue curve in Fig. 2(c) of main text. $v_c$ is very close to the sound velocity of an elongated Fermi gas $v_s' = \frac{\xi}{\sqrt{5}} v_F = 0.35 v_F$ (21). For comparison, in a nearly isotropic trap and a moving 1D lattice, the MIT group found a critical velocity $v_c = 0.32 v_F$ (33).

![Figure S3: Example of dipole oscillations of the $^7$Li BEC for a large initial amplitude (blue circles). The blue solid line is a fit to equation (7) from main text with a phenomenological damping rate $\gamma = 3.1 s^{-1}$.

**BEC mean-field and Lee-Huang-Yang limit**

Here we evaluate the frequency shift $\delta \omega_b/\omega_b$ given by

$$\frac{\delta \omega_b}{\omega_b} \simeq \frac{1}{2} g_{bf} \left( \frac{dn_i^{(0)}}{d\mu_f} \right)_{r=0},$$

(S4)

in the limit where the Fermi superfluid is a molecular BEC of composite Fermi-Fermi dimers. The dimers have a mass $m_d = 2m_f$ and a binding energy $E_d = \hbar^2/m_d a_d^2$, where $a_d = 0.6 a_f$ is the dimer-dimer scattering length (34). The Lee-Huang-Yang EoS for the molecular BEC reads

$$n_d = \frac{\mu_d}{g_d} \left( 1 - \frac{32}{3\sqrt{\pi}} \sqrt{\frac{\mu_d a_d^3}{g_d}} \right)$$

(S5)

where $n_d = n_f/2$ is the density of dimers, $\mu_d = 2\mu_f + E_d$ their chemical potential, and $g_{dd} = 4\pi \hbar^2 a_d/m_d$ the coupling constant for the dimer-dimer interaction. Then we have $\frac{d}{d\mu_f} = 2 \frac{d}{d\mu_d}$ and thus

$$\frac{dn_i^{(0)}}{d\mu_f} = 4 \frac{\mu_d a_d^3}{g_d} \left( 1 - \frac{16}{\sqrt{\pi}} \sqrt{\frac{\mu_d a_d^3}{g_d}} \right).$$

(S6)
This quantity must be evaluated in the center of the trap \((r = 0)\) to infer the frequency shift \((S4)\). The second term in \((S6)\) is of first order in \(\sqrt{n_d a_d}\). We then evaluate its argument in the mean-field approximation which gives the usual expression for the chemical potential of a BEC in a harmonic trap:

\[
(\mu_d)_{r=0} = \frac{\hbar \bar{\omega}_f}{2} \left( 15 N_d a_d \sqrt{\frac{m_d \bar{\omega}_f}{\hbar}} \right)^{2/5}.
\]  

(S7)

Using \((S7)\) and the expression of the Fermi wave-vector:

\[
k_F = \sqrt{\frac{m_d \bar{\omega}_f}{\hbar} (6 N_d)^{1/6}},
\]  

(S8)

with \(N_f = 2 N_d\), we can recast our expression for the frequency shift \((S6)\) in the universal units used in the main text (Eq. (10)):

\[
\left( \frac{\mu_d a_d^3}{g_d} \right)_{r=0} = \frac{1}{8 \pi} \left( \frac{5}{2} \right)^{2/5} (a_d k_F)^{12/5}
\]  

(S9)

\[
\left( \frac{dn_f^{(0)}}{d\mu_f} \right)_{r=0} \approx \frac{2 m_f}{0.6 \pi \hbar^2 a_f} \left( 1 - 1.172 (k_F a_f)^{6/5} \right)
\]  

(S10)

\[
\frac{\delta \omega_b}{\omega_b} k_F a_{bf} \approx 6.190 \frac{1}{k_F a_f} \left( 1 - 1.172 (k_F a_f)^{6/5} \right)
\]  

(S11)

This limit is shown in green in Fig. S4. The mean-field approximation (red curve in Fig. S4) corresponds to the first term in Eq. (S11).

Figure S4: Predicted frequency shift (blue line) over a broad range of \(1/k_F a_f\). The dashed blue line shows the ideal Fermi gas limit. On the BEC side the green line shows the Lee-Huang-Yang prediction \((S11)\) and the red line the mean-field prediction.
Derivation of the coupled oscillator model using the sum-rule approach

We describe the dynamics of the system by a Hamiltonian

$$\hat{H} = \sum_{i,\alpha} \left[ \frac{\hat{p}^2_{\alpha,i}}{2m_\alpha} \right] + U(r_{\alpha,i}), \quad (S12)$$

where $\alpha = b, f$ labels the isotopes, and $U$ describes the total (trap+interaction) potential energy of the cloud.

Consider the operators $\hat{F}_\alpha = \sum_{N_\alpha i=1}^{N_\alpha} \hat{z}_{\alpha,i}$, where $z_{\alpha,i}$ is the position along $z$ of the $i$-th atom of species $\alpha = b, f$ and take $\hat{F}(a_f, a_b) = \sum_\alpha a_\alpha \hat{F}_\alpha$ an excitation operator depending on two mixing coefficients $(a_\alpha)$.

We introduce the moments $S_p$ defined by

$$S_p = \sum_n (E_n - E_0)^p \left| \langle n | \hat{F} | 0 \rangle \right|^2,$$

where $|n\rangle$ and $E_n$ are the eigenvectors and the eigenvalues of the Hamiltonian $\hat{H}$ (by definition $|0\rangle$ is the ground state and $E_0$ is its energy). Using the Closure Relation and first order perturbation theory, $S_1$ and $S_{-1}$ can be calculated exactly and we have

$$S_1 = -\sum_\alpha \frac{\hbar^2}{m_\alpha} N_\alpha a_\alpha^2 \quad (S13)$$

$$S_{-1} = -\frac{1}{k} \sum_{\alpha,\beta} a_\alpha a_\beta N_\alpha \frac{\partial \langle z_\alpha \rangle}{\partial b_\beta} \quad (S14)$$

where $k$ is the restoring force of the axial magnetic trap and $\langle z_\alpha \rangle$ is the center of mass position of atoms $\alpha$ in the presence of a perturbing potential $-k \sum_\beta b_\beta \hat{F}_\alpha$ corresponding to a shift of the trapping potential of species $\beta$ by a distance $b_\beta$. $\langle z_\alpha \rangle$ satisfies two useful conditions. First, using Hellmann-Feynman’s theorem, the matrix $N_\alpha \partial_{b_\beta} \langle z_\alpha \rangle = \partial_{b_\beta} \langle \hat{F} \rangle$ is symmetric. Secondly, if we shift the two traps by the same quantity $b_\beta = b$, the center of mass of the two clouds move by $\langle z_\alpha \rangle = b$. Differentiating this constraint with respect to $b$ yields the condition $\sum_\beta \partial_{b_\beta} \langle z_\alpha \rangle = 1$.

Experimentally, we observe that only two modes are excited by the displacement of the trap center. We label $|n=1\rangle$ and $|n=2\rangle$ the corresponding modes and we take $\hbar \omega_n = E_n - E_0$, with, by convention, $\omega_1 \leq \omega_2$. We thus have for any set of mixing parameters $(a_f, a_b)$,

$$\hbar^2 \omega_1^2 \leq \frac{S_1}{S_{-1}} \leq \hbar^2 \omega_2^2. \quad (S15)$$

To find the values of the two frequencies $\omega_1$ and $\omega_2$, one thus simply has to find the extrema of $S_1/S_{-1}$ with respect to $a_f$ and $a_b$. Using the sum rules (S13) and (S14), we see that

$$\frac{S_1}{S_{-1}} = \hbar^2 k \frac{\sum_\alpha N_\alpha/m_\alpha a_\alpha^2}{\sum_{\alpha,\beta} N_\alpha a_\alpha a_\beta \frac{\partial \langle z_\alpha \rangle}{\partial b_\beta}} \quad (S16)$$

This expression can be formally simplified by taking $a'_\alpha = a_\alpha \sqrt{N_\alpha/m_\alpha}$ and $\psi = (a'_f, a'_b)$. We then have

$$\frac{S_1}{S_{-1}} = \hbar^2 k \frac{\langle \psi | \psi \rangle}{\langle \psi | M \psi \rangle}, \quad (S17)$$
where the scalar product is defined by $\langle \psi | \psi' \rangle = \sum_\alpha \psi_\alpha \psi'_\alpha$ and the effective-mass operator is given by

$$M_{\alpha\beta} = \sqrt{m_\alpha m_\beta} \frac{N_\alpha \partial \langle z_\alpha \rangle}{N_\beta \partial b_\beta}.$$  

(S18)

With these notations, the frequencies $\omega_{i=1,2}$ are given by $\omega_i = \sqrt{k/m_i}$, where $m_i$ is an eigenvalue of $M$.

In the weak-coupling limit, the cross-terms $\partial b_\beta \langle z_\alpha \rangle$ ($\alpha \neq \beta$) are small and using their symmetry properties, we can write $M$ as $M_0 + M_1$ with

$$M_0 = \begin{pmatrix} m_f & 0 \\ 0 & m_b \end{pmatrix}$$  

(S19)

$$M_1 = \begin{pmatrix} -m_f \frac{\partial \langle z_f \rangle}{\partial b_f} & \sqrt{m_f m_b} \frac{N_f}{N_b} \frac{\partial \langle z_b \rangle}{\partial b_f} \\ \sqrt{m_f m_b} \frac{N_b}{N_f} \frac{\partial \langle z_b \rangle}{\partial b_f} & -m_b \frac{\partial \langle z_b \rangle}{\partial b_f} \end{pmatrix}$$  

(S20)

Since the matrix $M$ is symmetric we can use the usual perturbation theory to calculate its eigenvalues and eigenvectors. We have to first order

$$\tilde{m}_1 = m_f \left( 1 - \frac{\partial \langle z_f \rangle}{\partial b_f} \right)$$  

(S21)

$$\tilde{m}_2 = m_b \left( 1 - \frac{\partial \langle z_b \rangle}{\partial b_f} \right)$$  

(S22)

Using the symmetry of $N_\alpha \partial b_\beta \langle z_\alpha \rangle$, we see that in the experimentally relevant limit $N_f \gg N_b$, we have $\partial b_\beta \langle z_b \rangle \gg \partial b_\beta \langle z_f \rangle$. Thus the frequency of $^6\text{Li}$ is essentially not affected by the coupling between the two species. To leading order, we can identify $\omega_1$ ($\omega_2$) with $\tilde{\omega}_f$ ($\tilde{\omega}_b$) and we have

$$\tilde{\omega}_f \simeq \omega_f$$  

(S23)

$$\tilde{\omega}_b \simeq \omega_b \left( 1 + \frac{1}{2} \frac{\partial \langle z_b \rangle}{\partial b_f} \right)$$  

(S24)

To calculate the frequency $\tilde{\omega}_b$ we need to know the crossed-susceptibility $\partial b_\beta \langle z_b \rangle$. Since this is in equilibrium quantity, we can calculate it using the local-density approximation. We then obtain

$$\frac{\partial \langle z_b \rangle}{\partial b_f} = \frac{k g_{bf}}{N_b} \int d^3r z^2 \left( \frac{\partial m_b}{\partial \mu_f} \right) \left( \frac{\partial m_b}{\partial \mu_b} \right)$$  

(S25)

In the limit $N_b \ll N_f$, the bosonic cloud is much smaller than the fermionic cloud. We can therefore approximate this expression by

$$\frac{\partial \langle z_b \rangle}{\partial b_f} \simeq \frac{k g_{bf}}{N_b} \left( \frac{\partial m_f}{\partial \mu_f} \right)_0 \int d^3r z^2 \left( \frac{\partial m_b}{\partial \mu_b} \right)$$  

(S26)

where the index zero indicates that the derivative is calculated at the center of the trap. The integral can be calculated exactly and we finally obtain

$$\frac{\partial \langle z_b \rangle}{\partial b_f} = g_{bf} \left( \frac{\partial m_f}{\partial \mu_f} \right)_0,$$  

(S27)
where we recover Eq. (2) from main text.

To get the dynamics of the system after the excitation, we need to calculate the eigenvectors of the matrix $\mathcal{M}$. Note $\psi'_i = (a'_i, a'_i, f, b'_i)$ the eigenvector associated to the eigenvalue $\omega_i$. Using once more first order perturbation theory, we have

$$\psi'_1 = \left( \frac{\sqrt{m_f m_b}}{m_f - m_b} \sqrt{\frac{N_f}{N_i}} \frac{\partial (z_b)}{\partial b_i} \right)$$

$$\psi'_2 = \left( \frac{\sqrt{m_f m_b}}{m_b - m_f} \sqrt{\frac{N_b}{N_i}} \frac{\partial (z_b)}{\partial b_i} \right)$$

(S28)

(S29)

from which we deduce the vectors $\psi_{i=1,2} = (a_i, a_i, f, b_i)$ giving the excitation operator $\hat{F}(a_i, f, a_i, b_i)$. More precisely

$$\psi_1 = \sqrt{\frac{m_f}{N_f}} \left( \frac{1}{m_f - m_b} \frac{\partial (z_b)}{\partial b_i} \right)$$

$$\psi_2 = \sqrt{\frac{m_b}{N_b}} \left( \frac{1}{m_b - m_f} \frac{N_b}{N_i} \frac{\partial (z_b)}{\partial b_i} \right)$$

(S30)

(S31)

Note $d$ the initial displacement of the two species and expand the initial condition $Z = (z_f(0), z_b(0)) = (d, d)$ over the basis $\{\psi_1, \psi_2\}$. Since by construction the operator $\hat{F}(a_i, f, a_i, b_i)$ excites solely the mode $\omega_i$ we must have at time $t$ $Z(t) = \sum_i c_i \cos(\omega_i t) \psi_i$ (we assume that the initial velocities are zero). After a straightforward calculation, we get

$$z_f(t) = d \left[ \frac{(1 - \varepsilon \rho \eta) \cos(\omega_f t) + \eta \rho (1 + \varepsilon) \cos(\omega_b t)}{1 + \varepsilon^2 \rho \eta} \right]$$

$$z_b(t) = d \left[ -\varepsilon (1 - \rho \eta) \cos(\omega_f t) + (1 + \varepsilon) \cos(\omega_b t) \right]$$

(S32)

(S33)

with $\rho = N_b/N_f$, $\varepsilon = m_b/(m_b - m_f) \partial_{b_i}(z_b)$ and $\eta = m_f/m_b$. In experimentally relevant situations, we have $\varepsilon \ll 1$, $\rho \ll 1$ and $\eta \approx 1$. We can thus approximate the previous equations by

$$z_f(t) \approx d \left[ (1 - \varepsilon \rho) \cos(\tilde{\omega}_f t) + \rho \varepsilon \cos(\tilde{\omega}_b t) \right]$$

$$z_b(t) \approx d \left[ -\varepsilon \cos(\tilde{\omega}_f t) + (1 + \varepsilon) \cos(\tilde{\omega}_b t) \right]$$

(S34)

(S35)

and where according to Eq. (S24), we can take

$$\varepsilon = \frac{2 m_b}{m_b - m_f} \left( \frac{\tilde{\omega}_b - \omega_b}{\omega_b} \right).$$

(S36)

References


