



Supplementary Materials for

Symmetry-Protected Topological Orders in Interacting Bosonic Systems

Xie Chen, Zheng-Cheng Gu, Zheng-Xin Liu, Xiao-Gang Wen*

*To whom correspondence should be addressed. E-mail: xwen@perimeterinstitute.ca

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Supplementary Materials

Section 1: Projective Representation

Matrices $u(g)$ form a projective representation of symmetry group G if

$$u(g_1)u(g_2) = \omega(g_1, g_2)u(g_1g_2), \quad g_1, g_2 \in G. \quad (\text{S1})$$

Here $\omega(g_1, g_2) \in U(1)$ and $\omega(g_1, g_2) \neq 1$, which is called the factor system of the projective representation. The factor system satisfies

$$\omega^{s(g_1)}(g_2, g_3)\omega(g_1, g_2g_3) = \omega(g_1, g_2)\omega(g_1g_2, g_3), \quad (\text{S2})$$

for all $g_1, g_2, g_3 \in G$, where $s(g_1) = 1$ if g_1 is unitary and $s(g_1) = -1$ if g_1 is anti-unitary. If $\omega(g_1, g_2) = 1, \forall g_1, g_2$, this reduces to the usual linear representation of G .

A different choice of pre-factor for the representation matrices $u'(g) = \beta(g)u(g)$ will lead to a different factor system $\omega'(g_1, g_2)$:

$$\omega'(g_1, g_2) = \frac{\beta(g_1)\beta^{s(g_1)}(g_2)}{\beta(g_1g_2)}\omega(g_1, g_2). \quad (\text{S3})$$

We regard $u'(g)$ and $u(g)$ that differ only by a pre-factor as equivalent projective representations and the corresponding factor systems $\omega'(g_1, g_2)$ and $\omega(g_1, g_2)$ as belonging to the same class ω .

Suppose that we have one projective representation $u_1(g)$ with factor system $\omega_1(g_1, g_2)$ of class ω_1 and another $u_2(g)$ with factor system $\omega_2(g_1, g_2)$ of class ω_2 , obviously $u_1(g) \otimes u_2(g)$ is a projective presentation with factor system $\omega_1(g_1, g_2)\omega_2(g_1, g_2)$. The corresponding class ω can be written as a sum $\omega_1 + \omega_2$. Under such an addition rule, the equivalence classes of factor systems form an Abelian group, which is called the second cohomology group of G and is denoted as $\mathcal{H}^2(G, U(1))$. The identity element $1 \in \mathcal{H}^2(G, U(1))$ is the class that corresponds to the linear representation of the group.

$\omega(g_1, g_2)$ can be equivalently expressed as $\nu_2(\tilde{g}_0, \tilde{g}_1, \tilde{g}_2)$ with correspondence

$$\omega(g_1, g_2) = \nu_2(E, g_1, g_1 g_2) \quad (\text{S4})$$

where E is the identity element in the symmetry group G . $\nu_2(g_0, g_1, g_2)$ satisfies

$$\nu_2^{s(g)}(g_0, g_1, g_2) = \nu_2(gg_0, gg_1, gg_2), \quad g \in G \quad (\text{S5})$$

and the condition Eq.S2 satisfied by $\omega(g_1, g_2)$ becomes

$$\frac{\nu_2(g_1, g_2, g_3)\nu_2(g_0, g_1, g_3)}{\nu_2(g_0, g_2, g_3)\nu_2(g_0, g_1, g_2)} = 1 \quad (\text{S6})$$

Similarly, $\beta(g_1)$ can be equivalently written as $\nu_1(E, g_1)$ where

$$\nu_1^{s(g)}(g_0, g_1) = \nu_1(gg_0, gg_1) \quad (\text{S7})$$

The equivalence relation Eq.S3 between $\omega(g_1, g_2)$'s then becomes

$$\nu_2'(g_0, g_1, g_2) = \frac{\nu_1(g_0, g_1)\nu_1(g_1, g_2)}{\nu_1(g_0, g_2)}\nu(g_0, g_1, g_2). \quad (\text{S8})$$

In the following we will show $M(g)$ defined in Eq.2 forms a projective representation of symmetry group G . It is easy to check that:

$$\begin{aligned} & M(g_1)M(g_2)|g_0\rangle \\ &= M(g_1)\nu_2^{s(g_2)}(g_0, g_2^{-1}g^*, g^*)|g_2g_0\rangle \\ &= \nu_2^{s(g_1)}(g_2g_0, g_1^{-1}g^*, g^*)\nu_2^{s(g_1g_2)}(g_0, g_2^{-1}g^*, g^*)|g_1g_2g_0\rangle \end{aligned} \quad (\text{S9})$$

with $g^* = E$. We note that in the last line, the sign factor $s(g_1g_2)$ arises because the complex conjugate K could have a nontrivial action on $\nu_2^{s(g_2)}(g_0, g_2^{-1}g^*, g^*)$ if g_1 is an antiunitary.

According to the 2-cocycle condition Eq.S6, we have:

$$\begin{aligned}
& M(g_1)M(g_2)|g_0\rangle \\
& =\nu_2(g^*, g_1g^*, g_1g_2g^*)\nu_2(g_1g_2g_0, g^*, g_1g_2g^*)|g_1g_2g_0\rangle \\
& =\nu_2(g^*, g_1g^*, g_1g_2g^*)\nu_2^{s(g_1g_2)}(g_0, (g_1g_2)^{-1}g^*, g^*)|g_1g_2g_0\rangle \\
& =\nu_2(E, g_1, g_1g_2)M(g_1g_2)|g_0\rangle \\
& =\omega_2(g_1, g_2)M(g_1g_2)|g_0\rangle \tag{S10}
\end{aligned}$$

Now it is clear that $M(g)$ defined in Eq.2 forms a projective representation of the symmetry group G . However, $M(g)$ is usually reducible. It can be reduced to a direct sum of several irreducible projective representations which belong to the same class. Only irreducible projective representations describe the edge states of general 1D SPT phases.

Also we can show that, after the duality transformation (described in Fig.3) on the wavefunction in Eq.6, symmetry acts as in Eq.2 on the edge degree of freedom in the ground state. Recall that before the transformation, symmetry acts as complex conjugate K together with a change of basis $|E\rangle \rightarrow |\mathcal{T}\rangle$, $|\mathcal{T}\rangle \rightarrow |E\rangle$ on each g_i . After expanding each g_i into h_i^r and h_{i+1}^l and performing the basis transform

$$|h_i^l, h_i^r\rangle' = \nu_2(h_i^l, h_i^r, g^*)|h_i^l, h_i^r\rangle = \nu_2(g_{i-1}, g_i, g^*)|h_i^l, h_i^r\rangle$$

time reversal symmetry acts separately on each site and is composed of complex conjugation K together with unitary transformation on the basis states

$$\begin{aligned}
M_{lr}(g)|h_i^l, h_i^r\rangle' & = \frac{\nu_2^{s(g)}(h_i^l, h_i^r, g^*)}{\nu_2(gh_i^l, gh_i^r, g^*)}|gh_i^l, gh_i^r\rangle' \\
& = \frac{\nu_2^{s(g)}(h_i^l, g^{-1}g^*, g^*)}{\nu_2^{s(g)}(h_i^r, g^{-1}g^*, g^*)}|gh_i^l, gh_i^r\rangle'
\end{aligned}$$

which factorizes on the two degrees of freedom and acts in the same way as in Eq.2 on the edge.

Section 2: Group cohomology

The above discussion on the factor system of a projective representation can be generalized which gives rise to a cohomology theory of groups. In this section, we will briefly describe the group cohomology theory.

For a group G , let M be a G -module, which is an abelian group (with multiplication operation) on which G acts compatibly with the multiplication operation (*ie* the abelian group structure) on M :

$$g \cdot (ab) = (g \cdot a)(g \cdot b), \quad g \in G, \quad a, b \in M. \quad (\text{S11})$$

For the cases studied in this paper, M is simply the $U(1)$ group and a a $U(1)$ phase. The multiplication operation ab is the usual multiplication of the $U(1)$ phases. The group action is trivial $g \cdot a = a$ ($g \in G, a \in U(1)$) if g is unitary and $g \cdot a = a^*$ if g is anti-unitary.

Let $\nu_n(g_0, \dots, g_n)$ be a function of $(n + 1)$ group elements whose value is in the G -module $U(1)$. In other words, $\nu_n : G^{(n+1)} \rightarrow U(1)$. ν_n satisfies

$$\nu_n^{s(g)}(g_0, g_1, \dots, g_n) = \nu_n(gg_0, gg_1, \dots, gg_n), g \in G \quad (\text{S12})$$

We will call such a map ν_n an n -cochain: Let $\mathcal{C}^n(G, U(1)) = \{\nu_n\}$ be the space of all n -cochains. Note that $\mathcal{C}^n(G, U(1))$ is an Abelian group under the function multiplication $\nu_n''(g_0, \dots, g_n) = \nu_n(g_0, \dots, g_n)\nu_n'(g_0, \dots, g_n)$. We define a map d_n from $\mathcal{C}^n[G, U(1)]$ to $\mathcal{C}^{n+1}[G, U(1)]$:

$$\begin{aligned} & (d_n \nu_n)(g_0, g_1, \dots, g_{n+1}) \\ &= \prod_{i=0}^{n+1} \nu_n^{(-1)^i}(g_0, \dots, g_{i-1}, g_{i+1}, \dots, g_{n+1}) \end{aligned} \quad (\text{S13})$$

Let

$$\mathcal{B}^n(G, U(1)) = \{\nu_n | \nu_n = d_{n-1} \nu_{n-1}, \nu_{n-1} \in \mathcal{C}^{n-1}(G, U(1))\}$$

and

$$\mathcal{Z}^n(G, U(1)) = \{\nu_n | d_n \nu_n = 1, \nu_n \in \mathcal{C}^n(G, U(1))\}$$

$\mathcal{B}^n(G, U(1))$ and $\mathcal{Z}^n(G, U(1))$ are also Abelian groups which satisfy $\mathcal{B}^n(G, U(1)) \subset \mathcal{Z}^n(G, U(1))$

where $\mathcal{B}^1(G, U(1)) \equiv \{1\}$. The n -cocycle of G is defined as

$$\mathcal{H}^n(G, U(1)) = \mathcal{Z}^n(G, U(1)) / \mathcal{B}^n(G, U(1)) \quad (\text{S14})$$

Let us discuss some simple cases. From

$$(d_1 \nu_1)(g_0, g_1, g_2) = \nu_1(g_0, g_1) \nu_1(g_1, g_2) / \nu_1(g_0, g_2)$$

we see that

$$\mathcal{Z}^1(G, U(1)) = \{\nu_1 | \nu_1(g_0, g_1) \nu_1(g_1, g_2) = \nu_1(g_0, g_2)\}.$$

Since $\mathcal{B}^1(G, U(1)) \equiv \{1\}$ is trivial, $\mathcal{H}^1(G, U(1)) = \mathcal{Z}^1(G, U(1))$. If we define $\alpha(g) = \nu_1(E, g)$, it is easy to see that the first-cocycle is related to the one-dimensional representations of the group G .

From

$$\begin{aligned} & (d_2 \nu_2)(g_0, g_1, g_2, g_3) \\ &= \nu_2(g_1, g_2, g_3) \nu_2(g_0, g_1, g_3) / \nu_2(g_0, g_2, g_3) \nu_2(g_0, g_1, g_2) \end{aligned}$$

we see that

$$\begin{aligned} \mathcal{Z}^2(G, U(1)) &= \{\nu_2 | \\ & \nu_2(g_1, g_2, g_3) \nu_2(g_0, g_1, g_3) = \nu_2(g_0, g_2, g_3) \nu_2(g_0, g_1, g_2)\}. \end{aligned}$$

and

$$\begin{aligned} \mathcal{B}^2(G, U(1)) &= \{\nu_2 | \\ & \nu_2(g_0, g_1, g_2) = \nu_1(g_0, g_1) \nu_1(g_1, g_2) / \nu_1(g_0, g_2)\}. \end{aligned}$$

The 2-cocycles $\mathcal{H}^2(G, U(1)) = \mathcal{Z}^2(G, U(1))/\mathcal{B}^2(G, U(1))$ classify the projective representations discussed in section 1.

From

$$\begin{aligned} & (d_3\nu_3)(g_0, g_1, g_2, g_3, g_4) \\ &= \frac{\nu_3(g_1, g_2, g_3, g_4)\nu_3(g_0, g_1, g_3, g_4)\nu_3(g_0, g_1, g_2, g_3)}{\nu_3(g_0, g_2, g_3, g_4)\nu_3(g_0, g_1, g_2, g_4)} \end{aligned} \quad (\text{S15})$$

we see that

$$\begin{aligned} \mathcal{Z}^3(G, U(1)) &= \{\nu_3 | \\ & \frac{\nu_3(g_1, g_2, g_3, g_4)\nu_3(g_0, g_1, g_3, g_4)\nu_3(g_0, g_1, g_2, g_3)}{\nu_3(g_0, g_2, g_3, g_4)\nu_3(g_0, g_1, g_2, g_4)} = 1\}. \end{aligned}$$

and

$$\begin{aligned} \mathcal{B}^3(G, U(1)) &= \{\nu_3 | \\ & \nu_3(g_0, g_1, g_2, g_3) = \frac{\nu_2(g_1, g_2, g_3)\nu_2(g_0, g_1, g_3)}{\nu_2(g_0, g_2, g_3)\nu_2(g_0, g_1, g_2)}\} \end{aligned}$$

which give us the 3-cocycle $\mathcal{H}^3(G, U(1)) = \mathcal{Z}^3(G, U(1))/\mathcal{B}^3(G, U(1))$.

Section 3: Relationship with the topological- θ term of the $O(3)$ non-linear sigma model

In the section, we will show that the amplitudes Eq.5 can be formally regarded as the discrete group and discrete space time generalization of topological θ term of non-linear sigma model. Compared to the continuous formulation of the path integral $Z = \int Dg e^{-\int d\mathbf{x} d\tau \mathcal{L}[g(\mathbf{x}, \tau)]}$ g_i corresponds to the field $g(\mathbf{x}, t)$ and $\sum_{\{g_i\}}$ corresponds to the path integral $\int Dg e^{-S(\{g_i\})}$ is the action-amplitude on the discretized space-time that corresponds to $e^{-\int d^d\mathbf{x} d\tau \mathcal{L}[g(\mathbf{x}, \tau)]}$ of the continuous formulation and $\nu_2^{s_{ijk}}(g_i, g_j, g_k)$ corresponds to the action-amplitude $e^{-\int_{(i,j,k)} d\mathbf{x} d\tau \mathcal{L}[g(\mathbf{x}, \tau)]}$ on a single triangle (i, j, k) .

We will first review that the Haldane phase (a non-trivial 1D SPT phase) is described by a 2π -quantized topological term in continuous non-linear σ -model. (26) However, such kind of 2π -quantized topological terms cannot describe more general 1D SPT phases. We argue that to describe SPT phases correctly, we must generalize the 2π -quantized topological terms to discrete space-time.

Before considering a spin-1 chain, let us first consider a $(0 + 1)$ D non-linear σ -model that describes a single spin, whose imaginary-time action is given by

$$S = \oint dt \frac{1}{2g} (\partial_t \mathbf{n}(t))^2 + i s \int_{D^2} dt d\xi \mathbf{n}(t, \xi) \cdot [\partial_t \mathbf{n}(t, \xi) \times \partial_\xi \mathbf{n}(t, \xi)] \quad (\text{S16})$$

where $\mathbf{n}(t)$ is a unit 3d vector and we have assumed that the time direction forms a circle. The second term is the Wess-Zumino-Witten (WZW) term. (22, 23) We note that the WZW term cannot be calculated from the field $\mathbf{n}(t)$ on the time-circle. We have to extend $\mathbf{n}(t)$ to a disk D^2 bounded by the time-circle: $\mathbf{n}(t) \rightarrow \mathbf{n}(t, \xi)$ (see Fig.S1). Then the WZW term can be calculated from $\mathbf{n}(t, \xi)$. When $2s$ is an integer, WZW terms from different extensions only differ by a multiple of $2i\pi$. So e^{-S} is determined by $\mathbf{n}(t)$ and is independent of how we extend $\mathbf{n}(t)$ to the disk D^2 .

The ground states of the above non-linear σ -model have $2s + 1$ fold degeneracy, which form the spin- s representation of $SO(3)$. The energy gap above the ground state approaches to infinity as $g \rightarrow \infty$. Thus a pure WZW term describes a pure spin- s spin.

To obtain the action for the $SO(3)$ symmetric antiferromagnetic spin-1 chain, we can assume that the spins \mathbf{S}_i are described by a smooth unit vector field $\mathbf{n}(x, t)$: $\mathbf{S}_i = (-)^i \mathbf{n}(ia, t)$ (see Fig. S2b). Putting the above single-spin action for different spins together, we obtain the following $(1+1)$ D non-linear σ -model (27)

$$S = \int dx dt \frac{1}{2g} (\partial \mathbf{n}(x, t))^2 + i\theta W, \quad \theta = 2\pi, \quad (\text{S17})$$

where $W = (4\pi)^{-1} \int dt dx \mathbf{n}(t, x) \cdot [\partial_t \mathbf{n}(t, x) \times \partial_x \mathbf{n}(t, x)]$ and $i\theta W$ is the topological term. (27) If the space-time manifold has no boundary, then $e^{-i\theta W} = 1$ when $\theta = 0 \bmod 2\pi$. We will call such a topological term – a 2π -quantized topological term. The above non-linear σ -model describes a gapped phase with short range correlation and the $SO(3)$ symmetry, which is the Haldane phase. (7) In the low energy limit, g flows to infinity and the fixed-point action contains only the 2π -quantized topological term. Such a non-linear σ -model will be called topological non-linear σ -model.

It appears that the 2π -quantized topological term has no contribution to the path integral and can be dropped. In fact, the 2π -quantized topological term has physical effects and cannot be dropped. On an open chain, the 2π -quantized topological term $2\pi iW$ becomes a WZW term for the boundary spin $\mathbf{n}_L(t) \equiv \mathbf{n}(x = L, t)$ (see Fig. S2). (26) The motion of \mathbf{n}_L is described by Eq.S16 with $s = 1/2$. So the Haldane phase of spin-1 chain has a spin-1/2 boundary spin at each chain end! (26, 28, 29)

We see that the Haldane phase is described by a fixed point action which is a topological non-linear σ -model containing only the 2π -quantized topological term. *The non-trivialness of the Haldane phase is encoded in the non trivially quantized topological term.* (26, 30) From this example, one might guess that various SPT phases can be classified by various topological non-linear σ -models, and thus by various 2π -quantized topological terms. But such a guess is not correct.

This is because the fixed-point action (the topological non-linear σ -model) describes a short-range-correlated state. Since the renormalized cut-off length scale of the fixed-point action is always larger than the correlation length, the field $\mathbf{n}(x, t)$ fluctuates strongly even at the cut-off length scale. Thus, the fixed-point action has no continuum limit, and must be defined on discrete space-time. On the other hand, in our fixed-point action, $\mathbf{n}(x, t)$ is assumed to be a continuous field in space-time. The very existence of the continuum 2π -quantized topological

term depends on the non-trivial mapping classes from the *continuous* space-time manifold T^2 to the *continuous* target space S^2 . It is not self consistent to use such a continuum topological term to describe the fixed-point action for the Haldane phase.

As a result, the continuum 2π -quantized topological terms fail to classify bosonic SPT phases. For example, different possible continuum 2π -quantized topological terms in Eq.S17 are labeled by integers, while the integer spin chain has only two gapped phases protected by spin rotation symmetry: all even-integer topological terms give rise to the trivial phase and all odd-integer topological terms give rise to the Haldane phase. Also non-trivial SPT phases may exist even when there is no continuum 2π -quantized topological terms (such as when the symmetry G is discrete).

However, the general idea of using fixed-point actions to classify SPT phases is still correct. But, to use 2π -quantized topological terms to describe bosonic SPT phases, we need to generalize them to discrete space-time. In the following, we will show that this indeed can be done, using the (1+1)D model (S17) as an example.

A discrete (1+1)D space-time is given by a branched triangulation (see Fig.2a). Since $S = \int dx dt L$, on triangulated space-time, we can rewrite

$$e^{-S} = \prod \nu^{s(i,j,k)}(\mathbf{n}_i, \mathbf{n}_j, \mathbf{n}_k),$$

$$\nu^{s(i,j,k)}(\mathbf{n}_i, \mathbf{n}_j, \mathbf{n}_k) = e^{-\int_{\Delta} dx dt L} \in U(1), \quad (\text{S18})$$

where $\int_{\Delta} dx dt L$ is the action on a single triangle. We see that, on discrete space-time, the action and the path integral are described by a 3-variable function $\nu(\mathbf{n}_i, \mathbf{n}_j, \mathbf{n}_k)$, which is called action amplitude. The $SO(3)$ symmetry requires that

$$\nu(g\mathbf{n}_i, g\mathbf{n}_j, g\mathbf{n}_k) = \nu(\mathbf{n}_i, \mathbf{n}_j, \mathbf{n}_k), \quad g \in SO(3). \quad (\text{S19})$$

In order to use the action amplitude $\nu^{s(i,j,k)}(\mathbf{n}_i, \mathbf{n}_j, \mathbf{n}_k)$ to describe a 2π -quantized topological term, we must have $\prod \nu^{s(i,j,k)}(\mathbf{n}_i, \mathbf{n}_j, \mathbf{n}_k) = 1$ on any sphere. This can be satisfied iff

$\prod \nu^{s(i,j,k)}(\mathbf{n}_i, \mathbf{n}_j, \mathbf{n}_k) = 1$ on a tetrahedron – the simplest discrete sphere (See Fig.2b):

$$\frac{\nu(\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3)\nu(\mathbf{n}_0, \mathbf{n}_1, \mathbf{n}_3)}{\nu(\mathbf{n}_0, \mathbf{n}_2, \mathbf{n}_3)\nu(\mathbf{n}_0, \mathbf{n}_1, \mathbf{n}_2)} = 1. \quad (\text{S20})$$

(Another way to define topological term on discretized space-time can be found in (31).)

A $\nu(\mathbf{n}_0, \mathbf{n}_1, \mathbf{n}_2)$ that satisfies Eq.S19 and Eq.S20 is called a 2-cocycle. If $\nu(\mathbf{n}_0, \mathbf{n}_1, \mathbf{n}_2)$ is a 2-cocycle, then

$$\nu'(\mathbf{n}_0, \mathbf{n}_1, \mathbf{n}_2) = \nu(\mathbf{n}_0, \mathbf{n}_1, \mathbf{n}_2) \frac{\mu(\mathbf{n}_1, \mathbf{n}_2)\mu(\mathbf{n}_0, \mathbf{n}_1)}{\mu(\mathbf{n}_0, \mathbf{n}_2)} \quad (\text{S21})$$

is also a 2-cocycle, for any $\mu(\mathbf{n}_0, \mathbf{n}_1)$ satisfying $\mu(g\mathbf{n}_0, g\mathbf{n}_1) = \mu(\mathbf{n}_0, \mathbf{n}_1)$, $g \in SO(3)$. Since $\nu(\mathbf{n}_0, \mathbf{n}_2, \mathbf{n}_3)$ and $\nu'(\mathbf{n}_0, \mathbf{n}_2, \mathbf{n}_3)$ can continuously deform into each other, they correspond to the same kind of 2π -quantized topological term. So we say that $\nu(\mathbf{n}_0, \mathbf{n}_2, \mathbf{n}_3)$ and $\nu'(\mathbf{n}_0, \mathbf{n}_2, \mathbf{n}_3)$ are equivalent. The equivalent classes of the 2-cocycles $\nu(\mathbf{n}_0, \mathbf{n}_2, \mathbf{n}_3)$ give us $\mathcal{H}^2(S^2, U(1))$ – the 2-cohomology group of sphere S^2 with $U(1)$ coefficient. $\mathcal{H}^2(S^2, U(1))$ classifies the 2π -quantized topological term in *discrete* space-time and with S^2 as the target space.

Does $\mathcal{H}^2(S^2, U(1))$ classify the SPT phases with $SO(3)$ symmetry? The answer is no. We know that S^2 is just one of the symmetric spaces of $SO(3)$. To classify the SPT phases, we need to replace the target space S^2 by the maximal symmetric space, which is the group itself $SO(3)$ (see (32) for more discussions). So we need to consider discrete non-linear σ -model described by $\nu(g_i, g_j, g_k)$, $g_i, g_j, g_k \in SO(3)$. Now the 2-cocycle conditions becomes

$$\begin{aligned} \nu(gg_i, gg_j, gg_k) &= \nu(g_i, g_j, g_k) \in U(1), \\ \frac{\nu(g_1, g_2, g_3)\nu(g_0, g_1, g_3)}{\nu(g_0, g_2, g_3)\nu(g_0, g_1, g_2)} &= 1, \end{aligned} \quad (\text{S22})$$

which defines a “group cohomology” $\mathcal{H}^2(SO(3), U(1))$. It classifies the 2π -quantized topological term for the maximal symmetric space. It also classifies the SPT phases with $SO(3)$ symmetry in (1+1)D.

Section 4: Branched triangulation and topological invariant amplitudes

As we have shown, the bosonic symmetry protected order in $d + 1$ D can be described by the amplitude:

$$Z = \frac{\sum_{\{g_i\}}}{|G|^{N_v}} \prod \nu_{d+1}^{s_{ij\dots k}}(g_i, g_j, \dots, g_k), \quad (\text{S23})$$

where $g_i, g_j, \dots, g_k \in G$ are group elements of the symmetry group G and $|G|$ is the order of G . The $d + 1$ -cocycle $\nu_{d+1}^{s_{ij\dots k}}(g_i, g_j, \dots, g_k)$ is defined on the branched $d + 1$ -simplex, with $s_{ij\dots k} = \pm 1$ uniquely determined by the orientation of the corresponding $d + 1$ -simplex. In the following, we will show they are topological invariant amplitudes. Hence, they represent a class of fixed point amplitudes of (symmetry protected) topologically ordered phases.

First, we need to give a branching structure to the discretized space-time. A branching is a choice of an orientation of each edge of an n -simplex such that there is no oriented loop on any triangle. For example, Fig.S3(a) is a branched 2-simplex and (c) is a branched 3-simplex. However, (b) is not an allowed branching because all its three edges contain the same orientations and thus form an oriented loop. (d) is also not allowed because one of its triangle contains an oriented loop. It is easy to check that any consistent branched structure can induce a canonical ordering for the vertices of n -simplex. Indeed, the branching structure will also induce an canonical orientation of the n -simplex, see Fig.S4.

To understand the geometric meaning of the topological invariance of the above amplitude, let us first discuss (1+1)D fixed-point action-amplitude with a symmetry group G . For a (1+1)D

system on a complex with a branching structure, a fixed-point action-amplitude (see Fig.S5)

$$\begin{aligned}
e^{-S(\{g_i\})} &= \prod_{\{ijk\}} \nu_2^{s_{ijk}}(g_i, g_j, g_k) \\
&= \nu_2^{-1}(g_1, g_2, g_3) \nu_2(g_0, g_4, g_3) \nu_2^{-1}(g_5, g_1, g_0) \times \\
&\quad \nu_2(g_1, g_0, g_3) \nu_2^{-1}(g_5, g_0, g_4)
\end{aligned} \tag{S24}$$

where each triangle contribute to a phase factor $\nu_2^{s_{ijk}}(g_i, g_j, g_k)$, $\prod_{\{ijk\}}$ multiply over all the triangles in the complex Fig.S5. Note that the first variable g_i in $\nu_2(g_i, g_j, g_k)$ correspond to the vertex with two out going edges, the second variable g_j to the vertex with one out going edges, and the third variable g_k to the vertex with no out going edges. $s_{ijk} = \pm 1$ depending on the orientation of $i \rightarrow j \rightarrow k$ to be anti-clock-wise or clock-wise.

In order for the action-amplitude to represent a quantized topological θ -term, we must choose $\nu_2(g_i, g_j, g_k)$ such that

$$e^{-S(\{g_i\})} = \prod_{\{ijk\}} \nu_2^{s_{ijk}}(g_i, g_j, g_k) = 1 \tag{S25}$$

on closed space-time complex without boundary, in particular, on a tetrahedron with four triangles (see Fig.2b):

$$\begin{aligned}
e^{-S(\{g_i\})} &= \prod_{\{ijk\}} \nu_2^{s_{ijk}}(g_i, g_j, g_k) \\
&= \frac{\nu_2(g_1, g_2, g_3) \nu_2(g_0, g_1, g_3)}{\nu_2(g_0, g_1, g_2) \nu_2(g_0, g_2, g_3)} = 1
\end{aligned} \tag{S26}$$

Also, in order for our system to have the symmetry generated by the group G , its action-amplitude must satisfy

$$\begin{aligned}
e^{-S(\{g_i\})} &= e^{-S(\{gg_i\})}, \text{ if } g \text{ contains no T} \\
\left(e^{-S(\{g_i\})}\right)^\dagger &= e^{-S(\{gg_i\})}, \text{ if } g \text{ contains one T}
\end{aligned} \tag{S27}$$

where T is the time-reversal transformation. This requires

$$\nu_2^{s(g)}(g_i, g_j, g_k) = \nu_2(gg_i, gg_j, gg_k). \quad (\text{S28})$$

Eq.S27 and Eq.S28 happen to be the conditions of 2-cocycles $\nu_2(g_0, g_1, g_2)$ of G . Thus the action-amplitude Eq.S24 constructed from a 2-cocycle $\nu_2(g_0, g_1, g_2)$ is a quantized topological θ -term.

If $\nu_2(g_0, g_2, g_3)$ satisfy Eq.S27 and Eq.S28, then

$$\nu'_2(g_0, g_2, g_3) = \nu_2(g_0, g_2, g_3) \frac{\mu_1(g_1, g_2)\mu_1(g_0, g_1)}{\mu_1(g_0, g_2)} \quad (\text{S29})$$

also satisfy Eq.S27 and Eq.S28, for any $\mu_1(g_0, g_1)$ satisfying $\mu_1(gg_0, gg_1) = \mu_1(g_0, g_1)$, $g \in G$. So $\nu'_2(g_0, g_2, g_3)$ also gives rise to a quantized topological θ -term. As we continuously deform $\mu_1(g_0, g_1)$, the two quantized topological θ -terms can be smoothly connected. So we say that the two quantized topological θ -terms obtained from $\nu_2(g_0, g_2, g_3)$ and $\nu'_2(g_0, g_2, g_3)$ are equivalent. We note that $\nu_2(g_0, g_2, g_3)$ and $\nu'_2(g_0, g_2, g_3)$ differ by a 2-coboundary $\frac{\mu_1(g_1, g_2)\mu_1(g_0, g_1)}{\mu_1(g_0, g_2)}$. So the equivalence classes of $\nu_2(g_0, g_2, g_3)$ is nothing but the cohomology group $\mathcal{H}^2(G, U_T(1))$. Therefore, the quantized topological θ -terms are classified by $\mathcal{H}^2(G, U_T(1))$.

We can also show that Eq.S24 is a fixed-point action-amplitude from the cocycle conditions on $\nu_2(g_i, g_j, g_k)$. The cocycle condition Eq.S26 induces two renormalization moves in the discretized manifold. Fig.S6 represents a $2 \leftrightarrow 2$ moves:

$$\nu_2(g_0, g_1, g_3)\nu_2(g_1, g_2, g_3) = \nu_2(g_0, g_1, g_2)\nu_2(g_0, g_2, g_3) \quad (\text{S30})$$

and Fig.S7 represents a $1 \leftrightarrow 3$ move:

$$\nu_2(g_1, g_2, g_3) = \nu_2(g_0, g_1, g_2)\nu_2(g_0, g_2, g_3)\nu_2^{-1}(g_0, g_1, g_3) \quad (\text{S31})$$

By using these two moves, different triangulation of the space-time can be mapped into each other without changing the action amplitude Eq.S23. Therefore, the action-amplitude is a fixed point form.

Geometrically, these two moves can be obtained by projecting a 3 simplex onto two two-dimensional surfaces. Fig.S8(a) gives the $2 \leftrightarrow 2$ move and Fig.S8(b) gives the $1 \leftrightarrow 3$ move. Note that the projection from opposite directions will induce opposite chiralities for the boundary of the tetrahedron, that's why we need to change the chiralities of the triangular in one side of basic moves. Such a change correspond to inverse ν_2 when we move it from left side to right side of the above equation, which is consistent with our rules of algebra. Similar process also works in higher dimensions. Different moves in d -dimension can be obtained from projecting a $d + 1$ -simplex in different ways. Such moves induces a renormalization flow in the discrete space-time under which the action-amplitude Eq.S23 is a fixed point.

A direct consequence of the the above renormalization moves is that the amplitude Eq.S23 is equal to 1 on any oriented closed $d + 1$ manifold. It is easy to see that the amplitude Eq.S23 is equal to 1 on any $d + 1$ -sphere. For example, the simplest triangulation of a 2-sphere is a tetrahedron(see Fig.2(b)), and the amplitude Eq.S23 is guaranteed to be 1 by the cocycle condition Eq.4. Next let us show that the Eq.S23 is equal to 1 on any oriented closed space-time surfaces. Instead of providing a rigorous mathematical proof, here we only present a simple proof for high-genus surfaces, which is good enough for all physical systems. To begin, we consider the simplest high-genus surface—a torus with a given triangulation. As seen in Fig.S9, by applying a so called surgery operation, which is implemented by cutting a thin ribbon(with physical degree of freedoms only lives on its boundary) on the torus and then gluing with two disks, we can deform a torus to a sphere. The key observation is that the amplitude Eq.S23 is unchanged under such an operation. To see this, let us use $W_t(g'_i, g'_j, g'_k, \dots)/W_b(g_i, g_j, g_k, \dots)$ to represent the amplitude of the top/bottom red/blue disk with boundary degree of freedoms $(g'_i, g'_j, g'_k, \dots)/(g_i, g_j, g_k, \dots)$ and use $V(g'_i, g'_j, g'_k, \dots; g_i, g_j, g_k, \dots)$ to represent the amplitude of the ribbon in between. Since the surface formed by the disks and the ribbon is topolog-

ically equivalent to a sphere, it is clear that its amplitudes is 1 and we have:

$$V(g'_i, g'_j, g'_k, \dots; g_i, g_j, g_k, \dots) W_t^{-1}(g'_i, g'_j, g'_k, \dots) W_b^{-1}(g_i, g_j, g_k, \dots) = 1 \quad (\text{S32})$$

Here we use W_t^{-1} and W_b^{-1} is because that the *inner* surfaces of the two disks and the ribbon form a closed oriented surface. Similar to the chirality choice in the proof of basic moves, the amplitudes for the inner surface should be W_t^{-1} and W_b^{-1} . Thus, we finally end up with:

$$V(g'_i, g'_j, g'_k, \dots; g_i, g_j, g_k, \dots) = W_t(g'_i, g'_j, g'_k, \dots) W_b(g_i, g_j, g_k, \dots) \quad (\text{S33})$$

, which implies the amplitude Eq.S23 is invariant under a surgery operation. It is easy to see that the above proof can be generalized into high genus surface and high dimensions.

The ground state wave-function can be obtained from the action amplitude on an open geometry as discussed in the main text

$$\Psi(\{g_i\}_M) = \prod_{\{i\dots j^*\}} \nu_{d+1}(g_i, \dots, g_j, g^*) \quad (\text{S34})$$

where $\{g_i\}_M$ is on M and g^* is inside M_{ext} (see Fig.2c). $\prod_{\{i\dots j^*\}}$ is the product over all simplices. An exactly soluble Hamiltonian can be constructed to realize this state as the gapped ground state. To see this, note that Ψ can be mapped through a local unitary transformation

$$U = \prod_{\{i\dots j^*\}} \nu_{d+1}^{-1}(g_i, \dots, g_j, g^*) |g_i\dots g_j\rangle \langle g_i\dots g_j| \quad (\text{S35})$$

to a total product state

$$\Psi_0(\{g_i\}_M) = 1 \quad (\text{S36})$$

As Ψ_0 is the gapped ground state of

$$H_0 = - \sum_k |\phi_k\rangle \langle \phi_k| \quad (\text{S37})$$

where $|\phi_k\rangle = \sum_{g_i \in G} |g_i\rangle$, Ψ is the gapped ground state of

$$H = U^\dagger H_0 U = \sum_k -U^\dagger |\phi_k\rangle \langle \phi_k| U \quad (\text{S38})$$

which is local and has symmetry G .

Figures:

Fig.S1: If we extend $\mathbf{n}(t)$ that traces out a loop to $\mathbf{n}(t, \xi)$ that covers the shaded disk, then the WZW term $\int_{D^2} dt d\xi \mathbf{n}(t, \xi) \cdot [\partial_t \mathbf{n}(t, \xi) \times \partial_\xi \mathbf{n}(t, \xi)]$ corresponds to the area of the disk.

Fig.S2: (a) The topological term W describes the number of times that $\mathbf{n}(x, t)$ wraps around the sphere (as we change t). (b) On an open chain $x \in [0, L]$, the topological term W in the (1+1)D bulk becomes the WZW term for the end spin $\mathbf{n}_L(t) = \mathbf{n}(L, t)$ (where the end spin at $x = 0$ is hold fixed).

Fig.S3: Examples of allowed((a),(c)) and unallowed((b),(d)) branching for 2-simplex and 3-simplex.

Fig.S4: (a): A branching structure of a n -simplex will induce a canonical ordering for its vertices. For example, if a 3-simplex contains a vertex with no incoming edge, then we can label this vertex as v^0 and canonically label the vertices of the rest 2-simplex as v^1, v^2, v^3 . Such a scheme can be applied for arbitrary n -simplex if n -simplex has a canonical label. (b): A branching structure will also induce a canonical orientation for the n -simplex. For example, we can use the right hand rule to determine the orientations of surfaces of the tetrahedron, then the orientation of the tetrahedron can be uniquely determined by the surface opposite to g^0 .

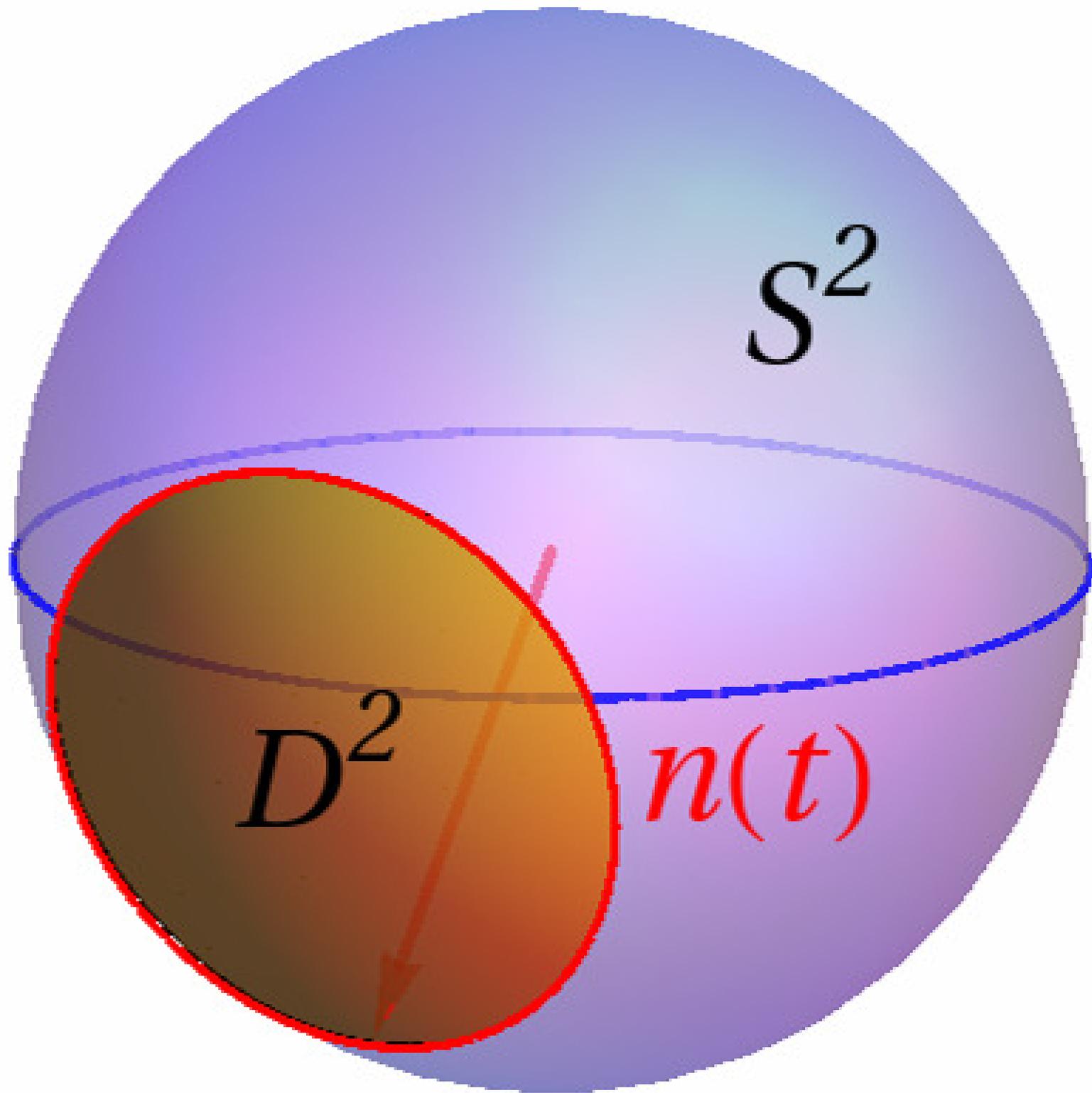
Fig.S5: The graphic representation of the action-amplitude $e^{-S(\{g_i\})}$ on a complex with a branching structure represented by the arrows on the edge. The vertices of the complex are labeled by i . Note that the arrows never for a loop on any triangle.

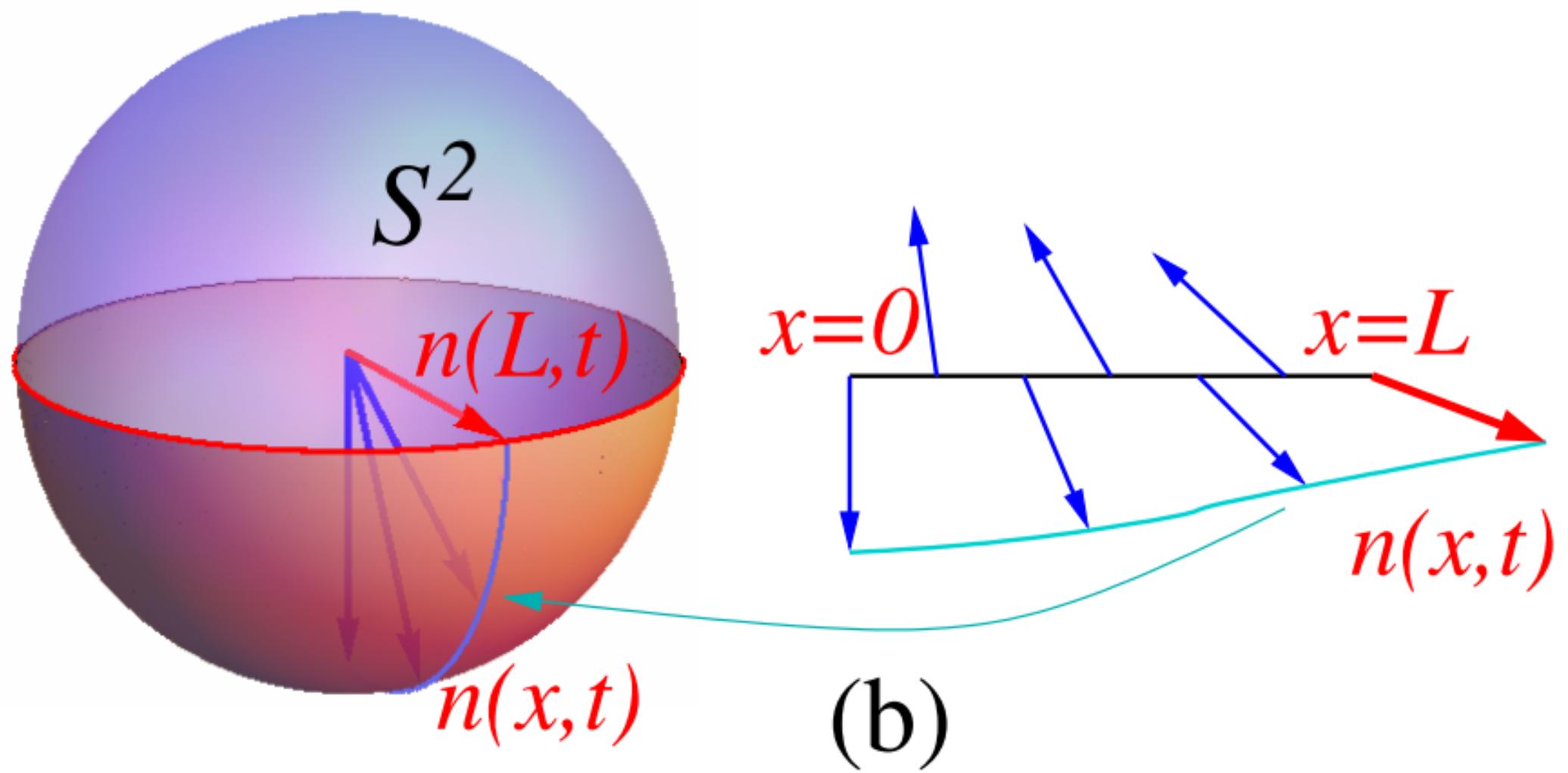
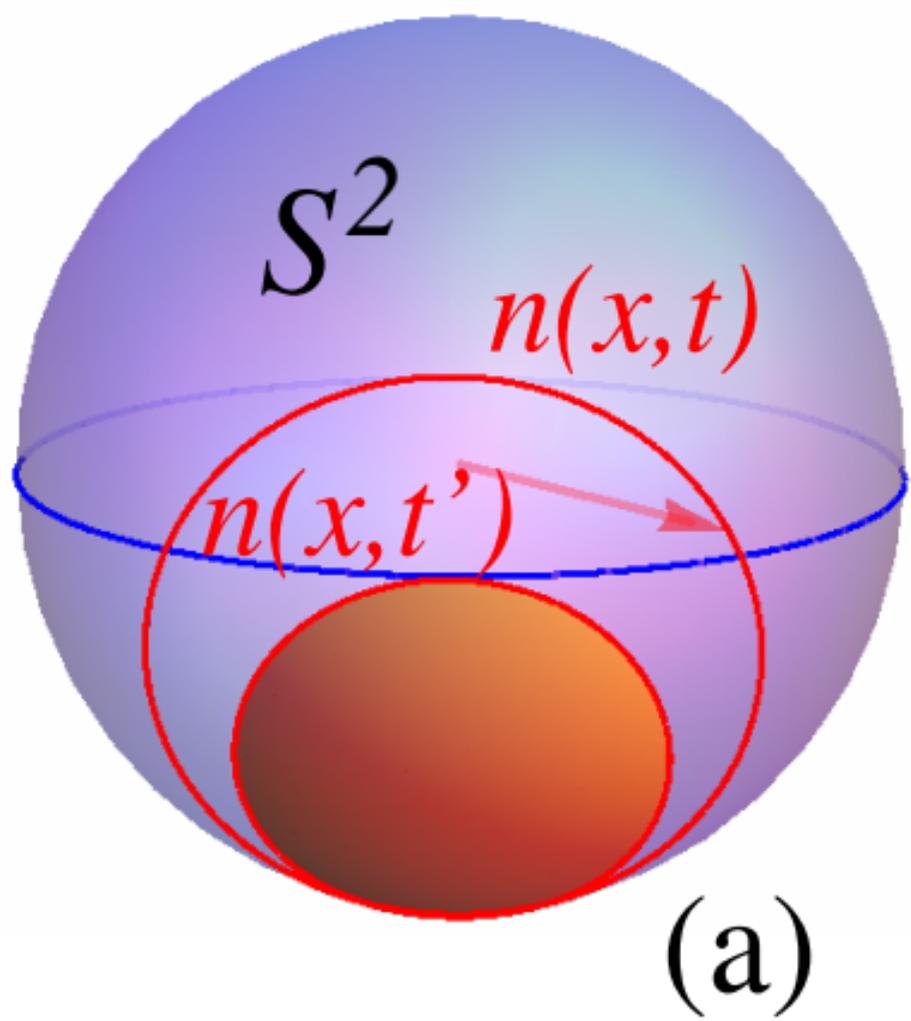
Fig.S6: Graphic representation of $\nu_2(g_0, g_1, g_2)\nu_2(g_0, g_2, g_3) = \nu_2(g_1, g_2, g_3)\nu_2(g_0, g_1, g_3)$. The arrows on the edges represent the branching structure.

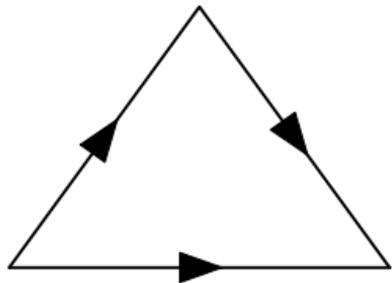
Fig.S7: Graphic representation of $\nu_2(g_1, g_2, g_3) = \nu_2(g_0, g_1, g_2)\nu_2(g_0, g_2, g_3)\nu_2^{-1}(g_0, g_1, g_3)$. The arrows on the edges represent the branching structure.

Fig.S8: $(d_2\nu_2)(g_0, g_1, g_2, g_3)$ can be represented as the boundary of a 3-simplex. (a) and (b) correspond to two different basic moves of 2-simplexes.

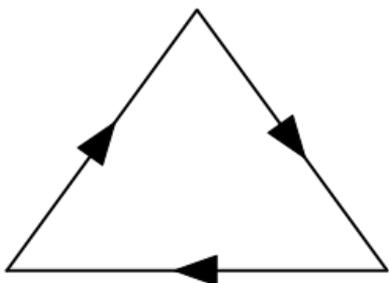
Fig.S9: By applying a surgery operation, which is implemented by cutting a ribbon on the torus and then gluing with two disks, we can deform a torus to a sphere.



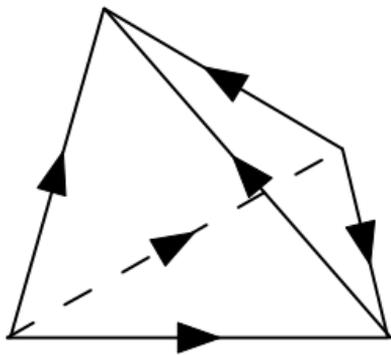




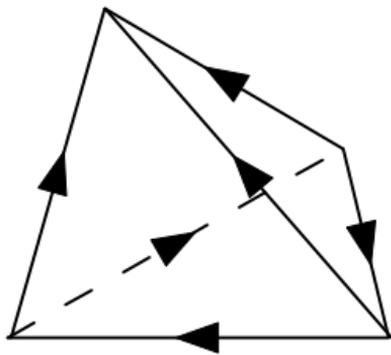
(a)



(b)

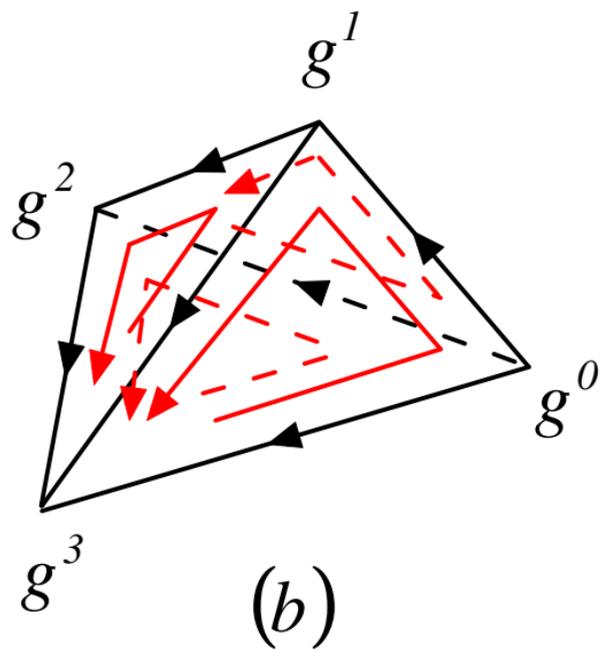
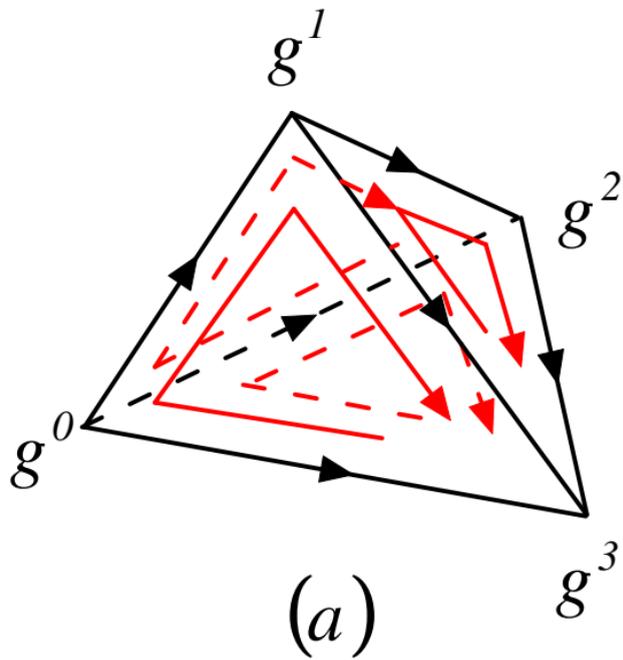


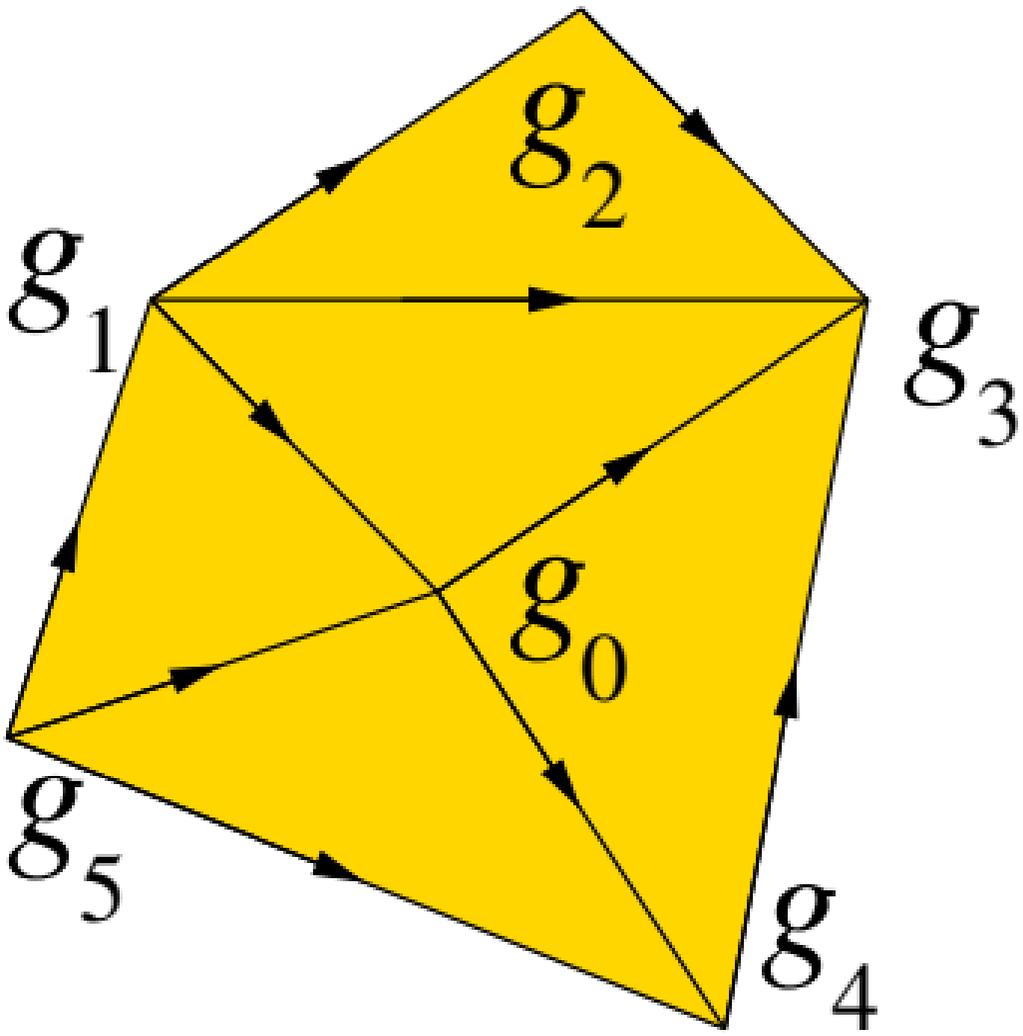
(c)

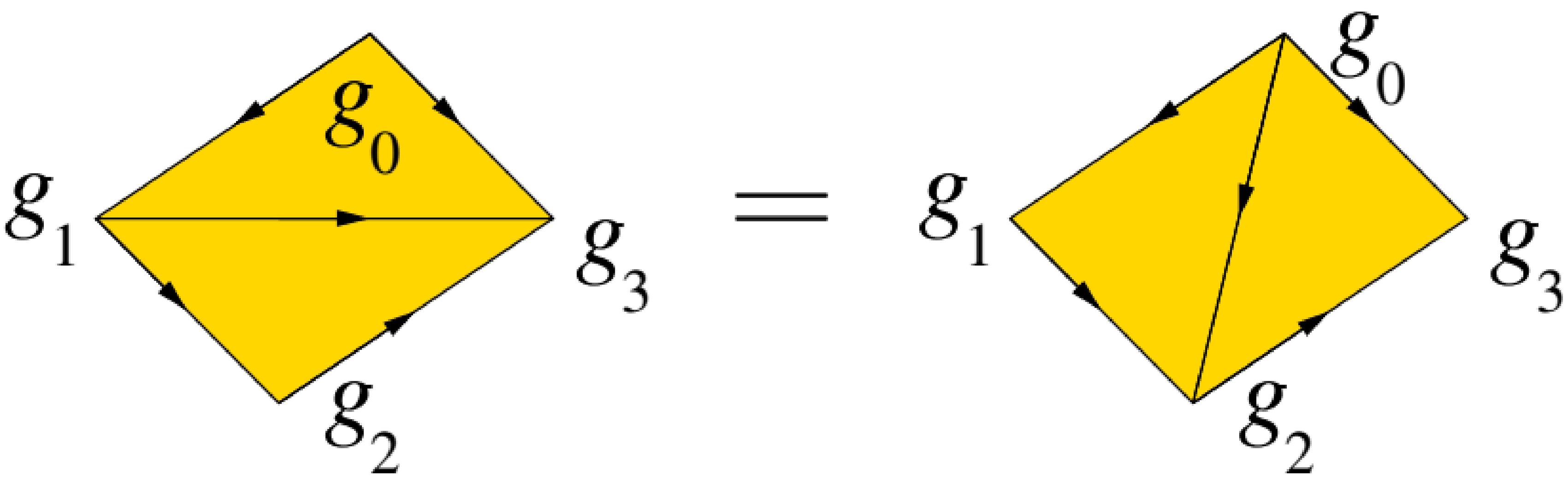


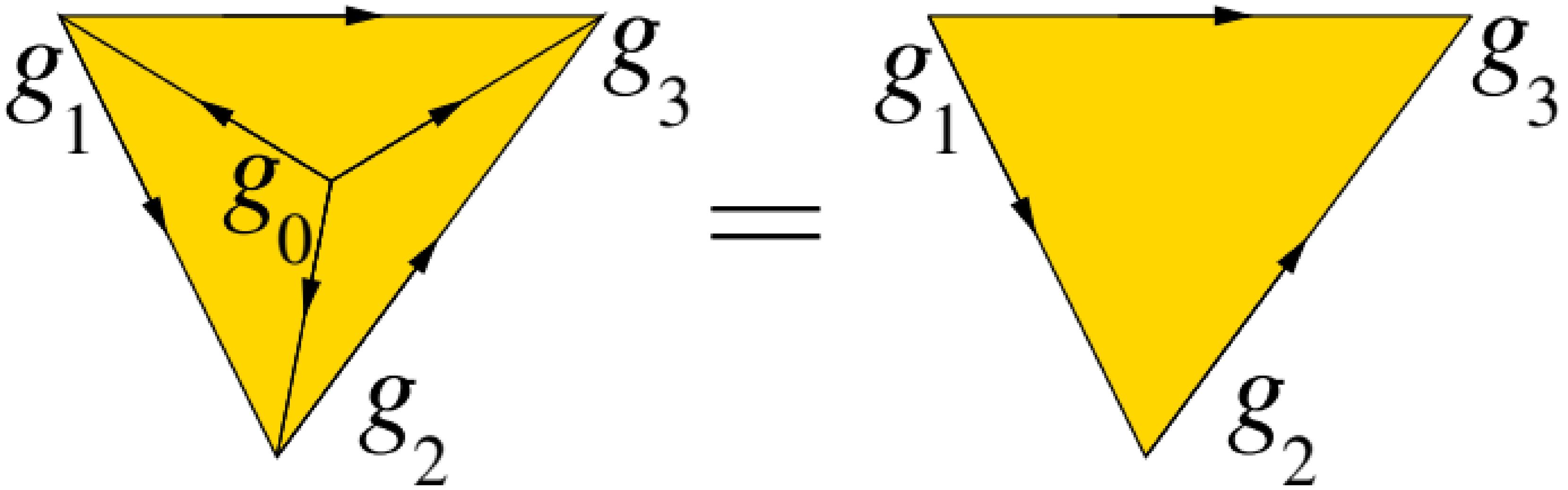
(d)

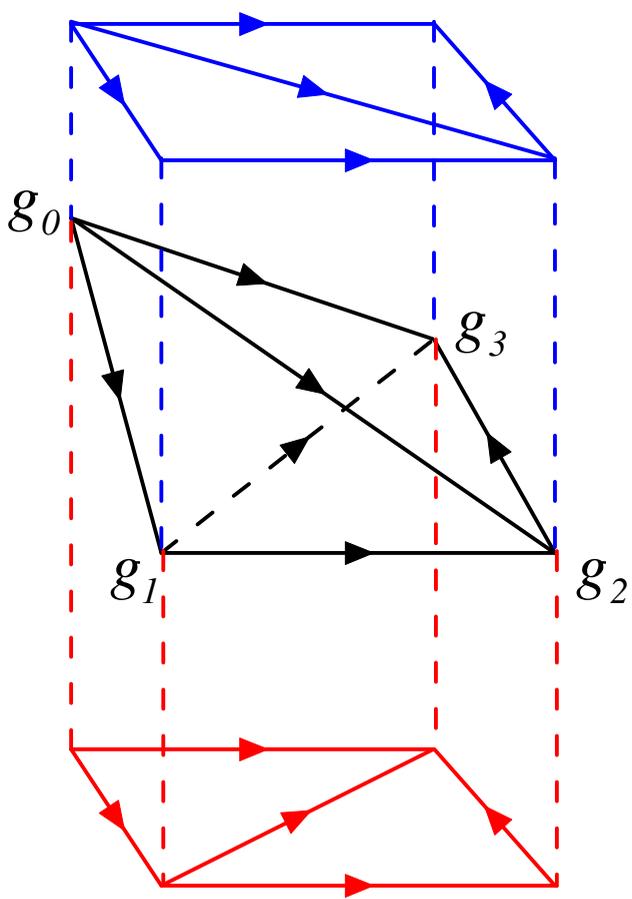
mirror



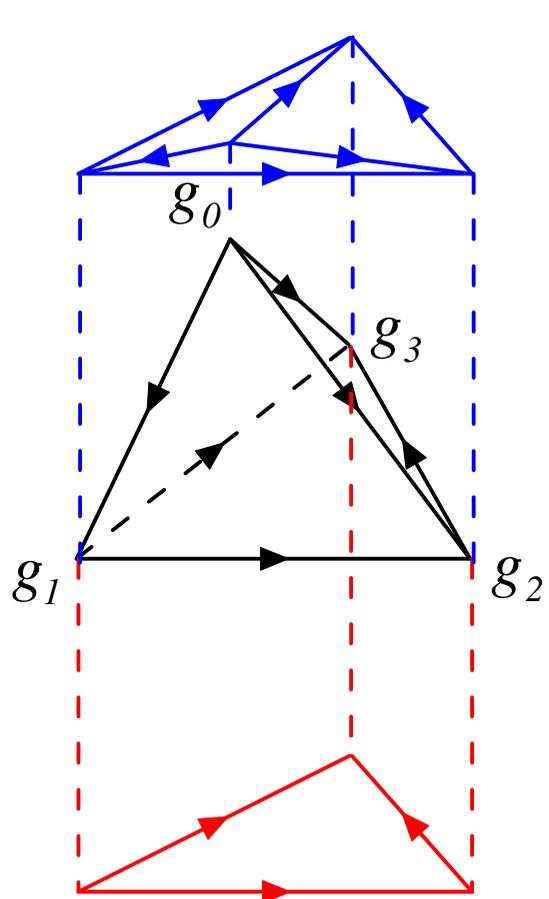




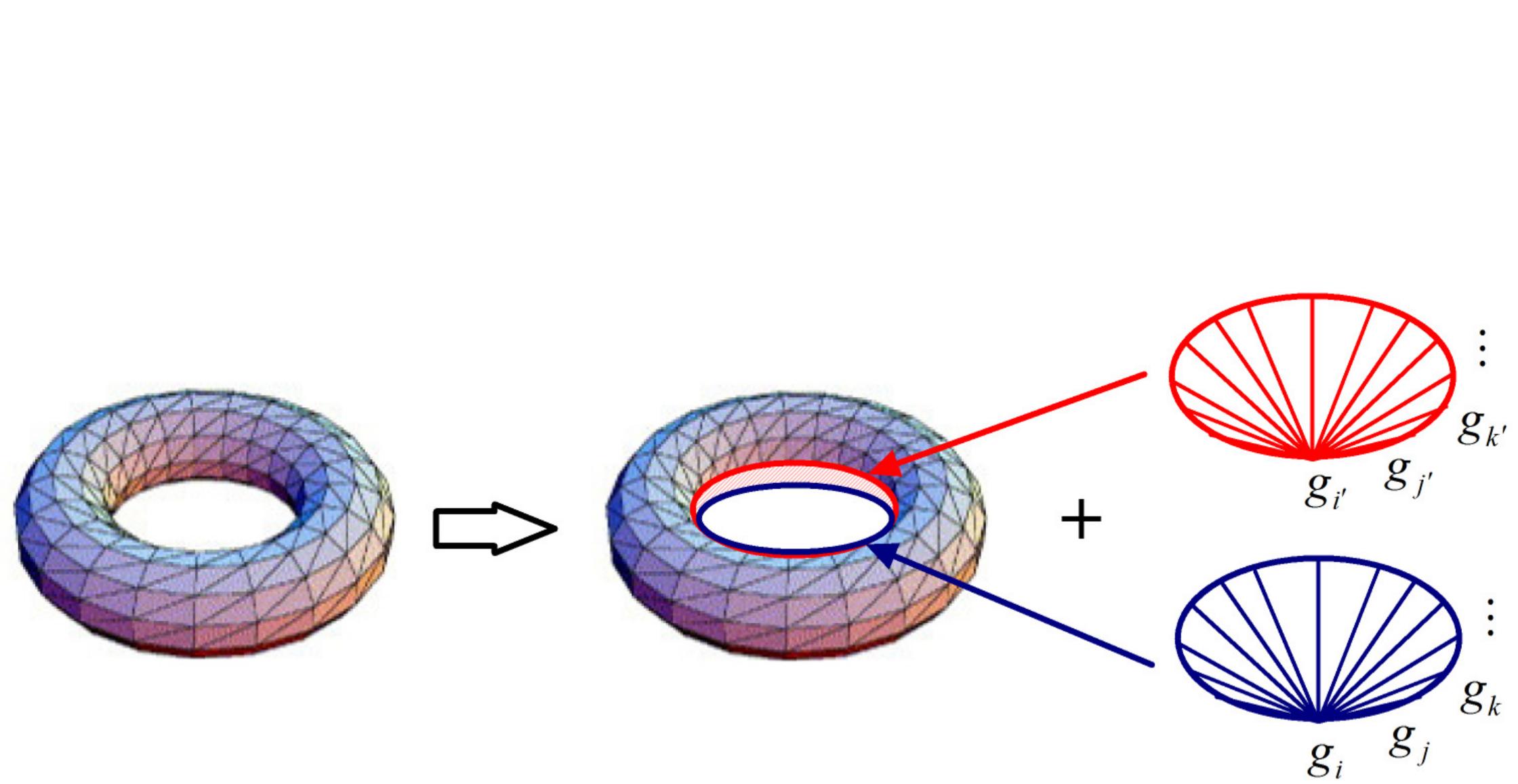




(a)



(b)



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